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Compensation phenomena in geometric partial differential equations

by

Benjamin Gilbert Sharp

Thesis

Submitted to The University of Warwick

for the degree of

Doctor of Philosophy

Mathematics Institute

25th June, 2012

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Declaration

I declare that the work in this thesis, is, to the best of my knowledge, original and my own work, except where otherwise indicated, cited, or commonly known. This thesis has not been submitted for a degree at another university. The work in chapters 4 and 5 (excluding Theorem 5.2.4 and section 4.3) is a joint work with Peter Topping and comprises a single paper [ST11] to appear in Transactions of the American Mathematical Society. The work from chapter 6 has been published on ArXiv [Sha11] but not yet submitted. The results in chapter 7 have not been submitted either online or to a journal as yet.

Abstract

In this thesis we present optimal and improved estimates for systems of critical elliptic PDE which arise as generalisations of natural geometric problems. We provide optimal regularity and compactness results for ‘Rivière’s equation’ for two dimensional domains via a new decay estimate, and we exhibit examples to show that the results are sharp. These results are presented in chapters 4 and 5. Such estimates generalise and improve known results in the classical setting.

In chapter 6 we improve the known regularity for the higher dimensional theory introduced by Rivière-Struwe leading to better estimates for solutions in this case. Such estimates in particular lead to an easy proof for the regularity for stationary harmonic maps.

We also present (in chapter 7) sharp results for a complex system of PDE, a consequence of which is a short proof of the full regularity for weakly harmonic maps from a Riemann surface into a closed Riemannian manifold.

Chapter 1

Introduction

Compensation phenomena here (broadly speaking) refers to regularity properties of certain non-linear PDE coming from geometry which, from a purely analytical perspective, have a poor regularity theory; however due to their geometric nature possess an improved structure (compensating for a lack of integrability) which allows for an analytical advantage. The main theme being that solutions u to such PDE *a-priori* take the form

$$-\Delta u = f \in L^1 \tag{1.1}$$

from which less than one might expect can be said of the regularity of u. For instance we cannot conclude from this that $\nabla^2 u \in L^1$, as opposed to the case where $f \in L^p$ for $1 < p < \infty$ (when one has $\nabla^2 u \in L^p$ by standard Calderon-Zygmund estimates). However it is known that when $f \in L^1$ has some extra structure, we can overcome this lack of integrability and conclude in particular that $\nabla^2 u \in L^1$. This apparently small concession leads to drastic improvements in the regularity theory of such solutions. A first example is in the case of weakly harmonic maps $u \in W^{1,2}(B_1, S^{m-1})$ where $B_1 \subset \mathbb{R}^2$ is the unit disc and $S^{m-1} \hookrightarrow \mathbb{R}^m$ is the round sphere. Such maps are critical points of the Dirichlet energy

$$E(u) = \frac{1}{2}\int_{B_1} |\mathrm{d}u|^2$$

and satisfy (in a weak sense)

$$-\Delta u = u|\nabla u|^2 \in L^1.$$

In this case it is known (due to Shatah [Sha88], see also [Hél02]) that $\mathrm{div}(u^i \nabla u^j - u^j \nabla u^i) = 0$ and we can write

$$-\Delta u^i = \sum_j (u^i \nabla u^j - u^j \nabla u^i).\nabla u^j$$

whence we have (see [CLMS93]) that the right hand side $\sum_j (u^i \nabla u^j - u^j \nabla u^i).\nabla u^j \in \mathscr{H}^1$, the Hardy space (see section 2.6 for definitions). The Hardy space is a strict subspace of L^1 and can be thought of as a replacement for L^1 in elliptic regularity theory. In particular we can conclude $u \in W^{2,1}_{loc} \hookrightarrow C^0$ in two dimensions. A classical result that continuous weakly harmonic maps are smooth then allows one to conclude full regularity. This structure was first noticed by Frédéric Hélein (see [Hél02]) but heavily relied upon the symmetries of the target (S^{m-1} is both homogeneous and isotropic) in order to conclude $\operatorname{div}(u^i \nabla u^j - u^j \nabla u^i) = 0$. This can be seen as being a result of Noether's theorem or alternatively that such Riemannian manifolds can be totally geodesically embedded into a Lie group (in this case $S^{m-1} \hookrightarrow SO(m)$) and thus we can lift a harmonic map $u : B_1 \to S^{m-1}$ into a harmonic map $[u] : B_1 \to SO(m)$. At this point $[u]$ satisfies $\operatorname{div}([u]^{-1}\mathrm{d}[u]) = 0$ which gives us our divergence free vector field. As alluded to above, this result has been generalised for all sufficiently symmetric targets in [Hél91b].

For a weakly harmonic map $u \in W^{1,2}(B_1 \subset \mathbb{R}^2, \mathscr{N})$, where $\mathscr{N} \hookrightarrow \mathbb{R}^m$ is isometrically embedded, $\mathscr{N}$ is in general not symmetric and we do not have an obvious divergence-free vector field. However a celebrated result of Hélein [Hél91a] tells us that weakly harmonic maps into *any* closed Riemannian manifold are still continuous (and therefore smooth). In this case Hélein showed that we can still re-write the PDE in order to capitalise on the appearance of terms lying in the Hardy space. This re-writing generally involves a subtle interplay between the highest order term of our PDE and the non-linearity or, to put it another way, we must re-write our PDE with respect to a different frame – a Coulomb gauge.

These techniques were employed respectively for higher dimensional domains by Evans [Eva91] (spherical targets) and Bethuel [Bet93] (general closed targets), allowing one to conclude that weakly harmonic maps are continuous (and therefore smooth) under the *added* assumption that $\nabla u \in M^{2,n-2}$ (a Morrey space, see section 2.8) and has sufficiently small norm in this space. This in particular allows one to conclude that weakly harmonic maps $u \in W^{1,n}(B_1, \mathscr{N})$ for $B_1 \subset \mathbb{R}^n$ are smooth. The extra assumption that $\nabla u \in M^{2,n-2}$ is necessary, since Tristan Rivière [Riv92] constructed weakly harmonic maps $u \in W^{1,2}(B_1, \mathscr{N})$ for $B_1 \subset \mathbb{R}^3$ that are 'nowhere regular'. Moreover the smallness condition on this norm is also necessary in order to conclude smoothness. This follows by considering the *energy minimising* map $u : B_1 \subset \mathbb{R}^3 \to S^2 \hookrightarrow \mathbb{R}^3$ defined by $u(x) := \frac{x}{|x|}$. This map is weakly harmonic and $\nabla u \in M^{2,n-2}$ however it is discontinuous at the origin.

The Morrey space $M^{2,n-2}$ is not 'picked out of thin air' in order to have a regularity theory, but actually arises naturally in the study of *weakly stationary* harmonic maps-where it is known (via the existence of a monotonicity formula) that $\|\nabla u\|_{M^{2,n-2}}$ can be made sufficiently small away from a closed singular set S with $\mathscr{H}^{n-2}(S) = 0$ ($\mathscr{H}^{n-2}$ is the $(n-2)$-dimensional Hausdorff measure). Thus the regularity result is that weakly stationary harmonic maps are smooth away from S. In the case of energy minimising harmonic maps, a classical result of Schoen-Uhlenbeck [SU82] tells us that $\dim_{\mathscr{H}}(S) \leq n-3$.

In 2007 Tristan Rivière [Riv07] generalised Hélein's regularity theory in two dimensions, allowing for an improved regularity theory for a much more general class of PDE. In particular he observed that critical points $u \in W^{1,2}(B_1, \mathbb{R}^m)$ to a general class of conformally invariant elliptic Lagrangians (quadratic in the gradient) solve the following (which we refer to as 'Rivière's equation')

$$-\Delta u = \omega . \nabla u \tag{1.2}$$

for $\omega \in L^2(B_1, so(m) \otimes \wedge^1 \mathrm{T}^*\mathbb{R}^2)$. This PDE is critical in the sense that *a-priori* it looks like (1.1). However the antisymmetry of ω and the appearance of ∇u in the right hand side allows one to change frame in an appropriate way, to conclude that $u \in W^{2,1}_{loc} \hookrightarrow C^0$ [Riv07, Theorem I.1]. This again comes down to the appearance of terms that are integrable, but also lie in $\mathscr{H}^1$. The frame of choice in this case is a small perturbation of the classical Coulomb frame (see Theorem 3.2.5). Moreover he shows that it is possible to write (1.2) in divergence form.

Michael Struwe and Tristan Rivière [RS08] showed that one can apply this theory suitably for higher dimensional domains. Here one considers $\omega \in M^{2,n-2}(B_1, so(m) \otimes \wedge^1 \mathrm{T}^*\mathbb{R}^n)$ and $\nabla u \in M^{2,n-2}$, where they prove that when $\|\omega\|_{M^{2,n-2}}$ is sufficiently small then u is Hölder continuous. This generalises the harmonic map regularity theory in higher dimensions and extends the theory in order to deal with less regular targets.

These perspectives isolate and generalise an important property of harmonic maps between Riemannian manifolds: That one can write such PDE in 'covariant divergence form'. We will see that one can interpret (1.2) in such a way, moreover in two dimensions it is possible to view harmonic maps as solving a covariant $\overline{\partial}$ type problem which also allows one to conclude the full regularity theory.

In this thesis we consider (1.2) in generality, and we prove optimal regularity results in dimension two (that $u \in W^{2,p}$ for all $p < 2$) and we improve the known regularity for higher dimensional domains, in particular we show that we can use the Hölder regularity already obtained by Rivière-Struwe in order to show that $\nabla u \in M^{q,n-2}$ for $q < \infty$ depending on the smallness of $\|\omega\|_{M^{2,n-2}}$.

We also consider a related first order complex problem and using Coulomb gauge techniques we give an optimal condition in which to conclude regularity. This leads to a new proof of the regularity theory for weakly harmonic maps in two dimensions.

1.1 Overview

In chapter 2 we give a background on some function spaces along with elliptic estimates thereon. We start by introducing a well-known convolution operator (the Newtonian potential) with a view to elliptic regularity theory on function spaces, therefore we will consider Riesz potentials and singular integral operators of Calderon-Zygmund type. We state the standard results in this area, however our later needs require us to study these operators on some exotic function spaces that lie on the 'border' of usual considerations (although in themselves form a perfectly standard and beautiful theory). Namely we study these operators on rearrangement invariant spaces (Lebesgue, Lorentz, Zygmund) but we also consider spaces that require more delicate analysis (Hardy, BMO, Morrey, Hölder, Campanato). We shall see that in the latter list there is some overlap; BMO, Morrey and Hölder spaces are subsets of the Campanato spaces.

One of the more striking aspects of the later theory presented in this thesis is that the study of these more exotic function spaces and their behaviour with respect to such operators is not a purely academic endeavour; more that these spaces naturally present themselves, and such an analysis is unavoidable for example in the regularity theory for weakly stationary harmonic maps. Fortunately this is a benefit (at least for the author) rather than a hinderance, as we hope to persuade the reader.

Chapter 3 is devoted to the use of the geometric set-up in order to improve our situation. We introduce some very basic notions from gauge theory on vector bundles. We will require a few existence results for changes of frame or gauge over real and complex vector bundles. It might seem strange to the reader that we are still using language from geometry rather than more 'plain speaking', however the natural setting for the gauge transforms we use is certainly in the realms of geometry and it is the only way I can see of justifying such operations. For instance the PDE we will consider naturally present connections on these vector bundles, the overriding theme being that we can express these connections (and therefore the PDE) with respect to a different frame. Therefore we try to find the frame which casts our PDE in the most favourable light (from a regularity perspective).

We also recall the set-up and known results in the regularity theory for weakly and weakly stationary harmonic maps, along with the recent generalisations and observations of Rivière and Rivière-Struwe for the re-writing of such PDE.

Chapters 4 and 5 contain the results of a joint work of myself with Peter Topping [ST11]. In chapter 4 we prove optimal regularity results for Rivière's equation for two dimensional domains via a new decay type estimate for the gradient, moreover we provide examples to show that these estimates are sharp. The techniques of this chapter rely on a perturbation of a Coulomb gauge discovered by Rivière (Theorem 3.2.5). We also prove regularity estimates for approximate solutions to (1.2) where the 'error term' is in the space $L\log L$ i.e. solutions to

$$-\Delta u = \omega.\nabla u + f \tag{1.3}$$

with $f \in L\log L$ – a space slightly smaller than L^1 but larger than L^p for $p > 1$.

In chapter 5 we prove a compactness result for a sequence of solutions to (1.3), with the error terms bounded in $L\log L$. This result is an improvement of a recent result of Li-Zhu [LZ09] where they use a slightly weaker version in order to obtain sharp results for sequences of maps into a round sphere with tension (see section 3.4) uniformly bounded in $L\log L$.

We also prove here (as a side issue) some compact embedding theorems for $L\log L$, using the continuous embedding of Trudinger [Tru67] for critical Sobolev exponents (Theorem 2.7.8).

In chapter 6 we prove some higher integrability results, extending the work of Rivière-Struwe [RS08] for the higher dimensional version of (1.2). Such results rely on the Hölder continuity already obtained [RS08] and also some improved Riesz potential estimates of Adams [Ada75] in the Morrey space setting. We actually require a small extension of the work of Adams (Proposition 2.8.4) to allow for an improvement for terms in the Hardy space $\mathscr{H}^1$. The results presented here lead, in particular, to a self contained proof of the regularity theory for stationary harmonic maps. Moreover these results only require use of Coulomb gauge methods and in particular provide a proof of the optimal regularity obtained in chapter 4 without the need to perturb the Coulomb gauge. The methods used from this chapter will be used (in a work in progress) to improve the known regularity for Dirac harmonic maps (joint with Miaomiao Zhu).

Chapter 7 is used in order to study a critical first order complex analogue of 1.2, where we consider $\alpha \in L^2(D, \mathbb{C}^m \otimes \wedge^{(1,0)}\mathrm{T}^*_{\mathbb{C}}D)$ solving

$$-\overline{\partial}\alpha = \omega^{\overline{z}} \wedge \alpha$$

and $\omega^{\overline{z}} \in L^2(D, gl(\mathbb{C}, m) \otimes \wedge^{(0,1)}\mathrm{T}^*_{\mathbb{C}}D)$ and $D \subset \mathbb{C}$ is the unit disc. We prove sharp results for the regularity theory of such PDE, showing that an added assumption is required for $\omega^{\overline{z}}$ in order to conclude optimal regularity of α. This result can be used in order to provide a relatively short proof of the regularity of weakly harmonic maps from a

Riemann surface into a closed Riemannian manifold.

Chapter 2

Analytic preliminaries – elliptic regularity theory

Here we will develop the tools necessary in order to prove local regularity for solutions to

$$-\Delta u = f \tag{2.1}$$

on some $U \subset \mathbb{R}^n$ and f belonging to some function space. We will also mention in passing the parallel theory for the first order complex operator $\overline{\partial}$ – i.e. we look at the regularity theory related to solutions of

$$\frac{\partial \alpha}{\partial \overline{z}} = g \tag{2.2}$$

except that in this case we of course restrict to domains $D \subset \mathbb{C}$.

2.1 The Newtonian potential

We require use of the fundamental solution of the Laplacian

$$\Gamma(x) := \begin{cases} -\frac{1}{2\pi}\log|x| & \text{if } n = 2 \\ \\ \frac{1}{(n-2)\omega_n}|x|^{2-n} & \text{if } n \geq 3, \end{cases}$$

where ω_n is the volume of the unit sphere $S^{n-1} \subset \mathbb{R}^n$. Notice that away from the origin Γ is smooth and solves $\Delta\Gamma = 0$. Since $\Gamma \in L^1_{loc}(\mathbb{R}^n)$ it defines a distribution and we can form the convolution with compactly supported smooth functions: Given $f \in C_c^\infty(\mathbb{R}^n)$

we define the Newtonian potential of f by

$$N[f] := \Gamma * f(x) = \int \Gamma(x-y) f(y)\,\mathrm{d}y.$$

Lemma 2.1.1. *Setting $w := N[f]$ we have that $w \in C^\infty(\mathbb{R}^n)$ and $-\Delta w = f$. The second assertion is equivalent to saying that (in a distributional sense) $-\Delta_x \Gamma(x-y) = \delta_y$ where δ_y is the Dirac mass centred at y.*

Proof. By a change of variables and the symmetry of Γ we may write

$$w(x) = \int \Gamma(z) f(x+z)\,\mathrm{d}z \tag{2.3}$$

which easily gives that w is continuous since $\Gamma \in L^1_{loc}$ and $f \in C^\infty_c$. Similarly we have

$$\frac{\partial^{|\alpha|} w}{\partial x^\alpha} = N\left[\frac{\partial^{|\alpha|} f}{\partial x^\alpha}\right]$$

for any multi index α (by the Lebesgue dominated convergence theorem) and hence that w is smooth.

Seeing the second assertion is also fundamental and requires applying Stokes' theorem. For some fixed x we can assume support$(f) \subset\subset B_R(x)$ and using (2.3) we have

$$\begin{aligned} -\Delta w(x) &= \int -\Gamma(z)\Delta_x f(x+z)\,\mathrm{d}z \\ &= \int_{B_R(x)\setminus B_\varepsilon(x)} -\Gamma(z)\Delta_z f(x+z)\,\mathrm{d}z + \int_{B_\varepsilon(x)} -\Gamma(z)\Delta_z f(x+z)\,\mathrm{d}z \\ &= \int_{B_R(x)\setminus B_\varepsilon(x)} -\Gamma(z)\Delta_z f(x+z)\,\mathrm{d}z + o(1) \end{aligned}$$

since $f(x+z)$ is symmetric in its variables and

$$\lim_{\varepsilon \downarrow 0} \int_{B_\varepsilon(x)} -\Gamma(z)\Delta_z f(x+z)\,\mathrm{d}z = 0$$

by absolute continuity of the Lebesgue integral.

Now we can apply Stokes' (or Green's) theorem on the domain $B_R(x)\setminus B_\varepsilon(x)$, the fact that $\Delta\Gamma = 0$ on this domain, and both $f, \frac{\partial f}{\partial x^i}$ vanish on $\partial B_R(x)$ to give

$$\int_{B_R(x)\setminus B_\varepsilon(x)} -\Gamma(z)\Delta_z f(x+z)\,\mathrm{d}z = \int_{\partial B_\varepsilon(x)} (\Gamma(z)\frac{\partial f}{\partial \nu_z}(x+z) - \frac{\partial \Gamma}{\partial \nu_z}(z) f(x+z))\,\mathrm{d}S^{n-1}_\varepsilon$$

where this is with respect to the outward pointing normal of $\partial B_\varepsilon(x)$.

It is easy to check that

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\left|\int_{\partial B_\varepsilon(x)}\Gamma(z)\frac{\partial f}{\partial \nu_z}(x+z)\,\mathrm{d}S_\varepsilon^{n-1}\right| &\le (\sup_{B_1(x)}|\nabla f|)\lim_{\varepsilon\downarrow 0}\begin{cases}|\varepsilon\log\varepsilon| & \text{if } n=2\\ \frac{\varepsilon}{n-2} & \text{if } n\ge 3\end{cases}\\ &= 0 \end{aligned}\tag{2.4}$$

and also

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\int_{\partial B_\varepsilon(x)}-\frac{\partial\Gamma}{\partial\nu_z}(z)f(x+z)\,\mathrm{d}S_\varepsilon^{n-1} &= \lim_{\varepsilon\downarrow 0}\int_{|z|=1}\frac{f(x+\varepsilon z)}{\omega_n}\,\mathrm{d}S^{n-1}\\ &= f(x)\end{aligned}$$

which proves the assertion.

□

In order to motivate the next section we will suppose $f\in L^p$ and see what we can infer about w. We will see that (for $f\in L^p$, $1\le p\le\infty$) $w=N[f]\in L^p_{loc}$ and we can write

$$\begin{aligned}\frac{\partial w}{\partial x^i}(x) &= \int\frac{\partial\Gamma}{\partial x^i}(x-y)f(y)\,\mathrm{d}y\\ &= \int -\frac{1}{\omega_n}\frac{x^i-y^i}{|x-y|^n}f(y)\,\mathrm{d}y\end{aligned}\tag{2.5}$$

and even (roughly speaking)

$$\begin{aligned}\frac{\partial^2 w}{\partial x^i\partial x^j}(x) &= \int\frac{\partial^2\Gamma}{\partial x^i\partial x^j}(x-y)f(y)\,\mathrm{d}y-\frac{\delta_{ij}}{n}f(x)\\ &= \int\frac{1}{\omega_n}\frac{n\frac{(x^i-y^i)(x^j-y^j)}{|x-y|^2}-\delta_{ij}}{|x-y|^n}f(y)\,\mathrm{d}y-\frac{\delta_{ij}}{n}f(x).\end{aligned}\tag{2.6}$$

We stress here that these expressions need justification – in particular the integrand in (2.6) is not absolutely integrable! We will justify whether or not these expressions make sense (and indeed in what sense) in the next section. For now though we can at least check that we have $-\Delta w=f$. We can understand $-\Delta w$ as a distribution, so pick $\phi\in C_c^\infty$ and compute

$$\begin{aligned}\int -w(x)\Delta\phi(x)\,\mathrm{d}x &= \int\int -\Gamma(x-y)f(y)\Delta\phi(x)\,\mathrm{d}y\,\mathrm{d}x\\ &= \int f(y)\int -\Gamma(x-y)\Delta\phi(x)\,\mathrm{d}x\,\mathrm{d}y\\ &= \int f(y)\phi(y)\,\mathrm{d}y\end{aligned}\tag{2.7}$$

where the second to last line follows from Lemma 2.1.1. Therefore we have proved that $-\Delta w = f \in L^p$ as distributions.

We will mention here that there is a convolution operator related to the $\overline{\partial}$ – problem (2.2), where an analogous theory holds – see section 2.5.

2.2 Riesz potentials and Riesz transforms: The L^p theory

2.2.1 Riesz potentials and related operators

We define convolution operators known as Riesz potentials I_α and related operators A_α, which crop up in harmonic analysis and in the theory of weakly differentiable functions. These operators are relevant to us since they enable one to estimate (derivatives of) the Newtonian potential of a function, and also allow one to prove sharp versions and generalisations of the Sobolev embedding theorems (see Remarks 2.2.2 and 2.8.3, Proposition 2.3.1 and Theorem 2.7.8).

Associated with A_α are kernels $a_\alpha \in C^\infty(\mathbb{R}^n \backslash 0)$ for $0 < \alpha < n$, which are homogenous of degree $\alpha - n$ and we assume $|a_\alpha| + |\nabla a_\alpha| \leq C(n,\alpha) < \infty$ on S^{n-1}. For instance when $n > 2$, Γ provides such an example. The term Riesz potential refers to a convolution operator I_α of a specific kernel $\frac{1}{\gamma(n,\alpha)} \frac{1}{|x|^{n-\alpha}}$ (see [Zie89] for a definition of $\gamma(n,\alpha)$), however we use the term in a more loose sense.

For $f \in L^1$ we write

$$A_\alpha[f](x) := a_\alpha * f(x) = \int a_\alpha(x-y) f(y)\, \mathrm{d}y,$$

and we observe that when $a_1^i(x) = -\frac{1}{\omega_n} \frac{x^i}{|x|^n}$ then A_1^i is an operator of the form (2.5).

Here is as good a place as any to introduce also the weak L^q space and the function $\lambda_f(s)$. We denote weak L^q by $L^{q,\infty}$. We will deal with these spaces in the context of Lorentz spaces in section 2.7, but for now we simply define them to consist of those functions for which there exists some C with

$$\lambda_f(s) := |\{x : |f(x)| > s\}| \leq C s^{-q}.$$

It is an easy exercise to check that this holds for $f \in L^q$ however if we consider the function

$$f(t) = \begin{cases} |t|^{-\frac{1}{q}} & \text{if } |t| < 1 \\ \\ 0 & \text{if } |t| \geq 1 \end{cases}$$

on $\mathbb{R}$ we see that $f \in L^{q,\infty}$ but $f \notin L^q$.

The proof of Theorem 2.2.1 will be left to section 2.8 where we will state a generalisation with respect to Morrey spaces – see Theorem 2.8.2.

Theorem 2.2.1. *Let $1 < p < \infty$ and $\alpha p < n$ (recall $0 < \alpha < n$), then $A_\alpha : L^p \to L^{\frac{np}{n-\alpha p}}$ is a bounded operator.*

When $p = 1$ and $\alpha < n$ we have $A_\alpha : L^1 \to L^{\frac{n}{n-\alpha},\infty}$ is bounded, i.e. there is some $C < \infty$ such that

$$|\{x : |A_\alpha[f](x)| > s\}| \leq C\|f\|_{L^1}^{\frac{n}{n-\alpha}} s^{-\frac{n}{n-\alpha}}.$$

Remark 2.2.2.

1. Note that whenever $U \subset \mathbb{R}^n$ has finite measure, we actually have that

$$\|A_\alpha[f]\|_{L^q(U)} \leq C\|f\|_{L^p}$$

for all $q \leq \frac{np}{n-\alpha p}$ and $\alpha p < n$ by Hölder's inequality. Moreover if $\alpha p \geq n$ then we can find $q < \frac{n}{\alpha} \leq p$ such that $\frac{nq}{n-\alpha q} = p$ and therefore $A_\alpha : L^p(U) \to L^p(U)$ is bounded for all p (remember $L^p(U) \subset L^q(U)$ for such U and $q < p$).

2. We will see below that for the Newtonian potential $w = N[f]$ we have

$$\frac{\partial w}{\partial x^i} = A_1^i[f],$$

(where A_1^i is defined above) i.e. (2.5) makes sense and we can conclude that w has partial derivatives that are in some Lebesgue space depending on how integrable f is.

3. It is easy to check that (see [Ste70] page 125) for general $g \in W^{1,p}(\mathbb{R}^n)$ we have

$$g(x) = \sum_{j=1}^{n} A_1^j\left[\frac{\partial g}{\partial x^j}\right](x),$$

thus the Sobolev embedding theorem in the case $1 < p < n$ can be recovered from Theorem 2.2.1 – this is exactly Sobolev's original proof. In fact, more generally there are Riesz potentials A_k such that if $g \in W^{k,p}$ then

$$g(x) = A_k[\nabla^k g](x).$$

This follows by an induction argument and standard properties of the Riesz potentials (see [Ste70] Chapter V). Thus the more general Sobolev embedding theorem for $kp < n$ can be recovered.

2.2.2 Riesz transforms

The singular integral operators we consider T, are also convolution operators, but whose kernels are 'singular'. Notice that for the Riesz potentials, the kernels are at least locally integrable, which is not the case here. The kernels associated to T are of the form $\frac{h(x)}{|x|^n}$ where h is homogeneous, of zero degree, has zero mean value on the sphere, and for ease of presentation is smooth (as a function defined on the sphere). Thus $h(kx) = h(x)$ for all $x \in \mathbb{R}^n \setminus \{0\}$, $k > 0$ and $\int_{S^{n-1}} h = 0$. We check that the integral in (2.6) is an operator of this form with

$$h_{ij}(x) = \frac{1}{\omega_n}\left(n\frac{x^i x^j}{|x|^2} - \delta_{ij}\right).$$

One must check that

$$\int_{S^{n-1}} \frac{n}{\omega_n} x^i x^j = \delta_{ij} \tag{2.8}$$

which follows when $i \neq j$ since this function is odd and we also have (for all i, j) $\int_{S^{n-1}} (x^i)^2 = \int_{S^{n-1}} (x^j)^2 = \frac{1}{n}\int_{S^{n-1}} |x|^2 = \frac{\omega_n}{n}$.

Let T_{ij} be the singular convolution operator related to h_{ij}. Far more important in harmonic analysis are the Riesz transforms R_i whose kernels (up to a constant) are given by

$$h_i(x) = \frac{x^i}{|x|}.$$

For a smooth function with compact support it is possible to show using the Fourier transform (see [Ste70]) that

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = -R_i[R_j[\Delta f]].$$

Thus if one can show boundedness of the Riesz transforms on a function space $R_i : X \to X$ and $\Delta f \in X$ then we have $\nabla^2 f \in X$ with estimates (for smooth functions). When $X = L^2$ this can be seen directly by integration by parts! Letting $f \in C_c^\infty$ we have

$$\begin{aligned}
\int |\nabla^2 f|^2 &= \int \sum_{i,j} \left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)^2 \\
&= -\int \sum_{i,j} \frac{\partial f}{\partial x^i} \frac{\partial^3 f}{\partial x^i \partial x^j \partial x^j} \\
&= \int \sum_{i,j} \frac{\partial^2 f}{\partial (x^i)^2} \frac{\partial^2 f}{\partial (x^j)^2} \\
&= \int |\Delta f|^2.
\end{aligned}$$

This also holds (by approximation) for $f \in W^{2,2}(\mathbb{R}^n)$.

We will use the singular integrals T_{ij} to do this job and in fact we will see that for the Newtonian potential $w = N[f]$

$$\frac{\partial^2 w}{\partial x^i \partial x^j} = T_{ij}[f] - \frac{\delta_{ij}}{n} f$$

holds in a suitable sense for appropriate f.

To this end we need to discuss the validity of such operators. For $f \in L^p$ we would like to define

$$T[f](x) := \frac{h(\cdot)}{|\cdot|^n} * f(x) = \int \frac{h(x-y)}{|x-y|^n} f(y)\,\mathrm{d}y,$$

however, since the kernel is not integrable this has to be made sense of using the principle value formula. We define

$$T^\varepsilon[f](x) \quad := \quad \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{h(x-y)}{|x-y|^n} f(y)\,\mathrm{d}y$$

then we set $T[f](x) := \lim_{\varepsilon \downarrow 0} T^\varepsilon[f](x)$ if this limit exists. Notice that for each $\varepsilon > 0$, $T^\varepsilon[f](x)$ is finite for $f \in L^p$ and $p < \infty$ by Hölder's inequality.

We state the theorem below as follows from Theorem 3, chapter II in [Ste70]. We will not prove it here but we strongly recommend reading the theory as it appears in [Ste70].

Theorem 2.2.3. *Let $f \in L^p$, $1 \le p < \infty$, then*

1. $\lim_{\varepsilon \to 0} T^\varepsilon[f](x) := T[f](x)$ *exists for almost every x.*

2. *The mapping $T : L^1 \to L^{1,\infty}$ is bounded.*

3. *For $1 < p < \infty$ the mapping $T : L^p \to L^p$ is bounded. Moreover $\lim_{\varepsilon \to 0} T^\varepsilon[f]$ converges in L^p-norm.*

Remark 2.2.4.

1. This theorem follows from a more general statement about singular convolution operators, which can also be found in [Ste70].

2. In particular this theorem holds for the Riesz transforms R_i and the operators T_{ij}.

3. We can also conclude (using the Marcikiewicz interpolation theorem-[Ste70]) that the bound $C(p,n,h)$ for which

$$\|T[f]\|_{L^p} \le C(p,n,h)\|f\|_{L^p}$$

can be given in terms of

$$C(p,n,h) = C(n,h)\begin{cases} \frac{p}{p-1} & \text{if } 1 < p \le 2 \\ p & \text{if } 2 < p < \infty. \end{cases}$$

2.3 Derivatives of the Newtonian potential

Equipped with Theorems 2.2.1 and 2.2.3 we can deal with weak derivatives of the Newtonian potential $w := N[f]$ for $f \in L^p$. By Remark 2.2.2 we have that that $N : L^p(U) \to L^p(U)$ is bounded for all bounded $U \subset \mathbb{R}^n$.

Proposition 2.3.1. *Suppose $f \in L^p(U)$ and $U \subset \mathbb{R}^n$ is some bounded domain, then upon extending f by zero,*

1. *when $1 < p < \infty$, $w \in W^{2,p}(U)$ with $\frac{\partial w}{\partial x^i} = A_1^i[f]$ and $\frac{\partial^2 w}{\partial x^i \partial x^j} = T_{ij}[f] - \frac{\delta_{ij}}{n} f$. In other words $N : L^p \to W^{2,p}(U)$ is bounded with*

$$\|w\|_{L^p(U)} + \|\nabla w\|_{L^p(U)} + \|\nabla^2 w\|_{L^p(U)} \le C\|f\|_{L^p(U)}.$$

2. *When $p = 1$, $w \in W^{1,(\frac{n}{n-1},\infty)}$ and $\frac{\partial w}{\partial x^i} = A_1^i[f]$. Or $N : L^1 \to W^{1,(\frac{n}{n-1},\infty)}(U)$ is bounded with*

$$\|w\|_{L^{(\frac{n}{n-1},\infty)}(U)} + \|\nabla w\|_{L^{(\frac{n}{n-1},\infty)}(U)} \le C\|f\|_{L^1(U)}.$$

In the above $C = C(p,n,U)$.

We have used the notation $W^{k,(p,\infty)}$ to denote the Sobolev space of functions whose k^{th} weak derivatives lie in $L^{p,\infty}$.

Remark 2.3.2. From now on we will also use the notation $N, \nabla N, \nabla^2 N$ to denote the obvious linear operators. For instance boundedness of T_{ij} implies boundedness of $\nabla^2 N$.

Proof. We approximate f by a sequence $\{f_l\} \subset C_c^\infty$ with $f_l \to f$ in L^p. Letting $w_l = N[f_l]$ we see that $\{w_l\} \subset C^\infty$ and we write

$$\begin{aligned} \frac{\partial w_l}{\partial x^i}(x) &= \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Gamma(z) \frac{\partial f_l}{\partial x^i}(x+z)\, \mathrm{d}z + \int_{B_\varepsilon(0)} \Gamma(z) \frac{\partial f_l}{\partial x^i}(x+z)\, \mathrm{d}z \\ &= \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Gamma(z) \frac{\partial f_l}{\partial z^i}(x+z)\, \mathrm{d}z + o(1) \\ &= \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} -\frac{\partial \Gamma}{\partial z^i}(z) f_l(x+z)\, \mathrm{d}z + o(1) \\ &=_{\lim_{\varepsilon\to 0}} \int \frac{1}{\omega_n} \frac{z^i}{|z|^n} f_l(x+z)\, \mathrm{d}z = A_1^i[f_l](x) \end{aligned} \tag{2.9}$$

for $a_1^i(z) = \frac{1}{\omega_n}\frac{z^i}{|z|^n}$. Similarly we have

$$\begin{aligned}
\frac{\partial^2 w_l}{\partial x^i \partial x^j}(x) &= \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Gamma(z) \frac{\partial^2 f_l}{\partial x^i \partial x^j}(x+z)\,\mathrm{d}z + \int_{B_\varepsilon(0)} \Gamma(z)\frac{\partial^2 f_l}{\partial x^i \partial x^j}(x+z)\,\mathrm{d}z \\
&= \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} -\frac{\partial \Gamma}{\partial z^i}(z)\frac{\partial f_l}{\partial z^j}(x+z) + \\
&\quad + \int_{\partial B_\varepsilon(0)} -\Gamma(z)\frac{\partial f}{\partial z^j}(z)\nu^i \mathrm{d}S_\varepsilon^{n-1}(z) + o(1).
\end{aligned}$$

We have already seen that (see (2.4))

$$\int_{\partial B_\varepsilon(0)} \Gamma(z)\frac{\partial f}{\partial z^j}(z)\nu^i \mathrm{d}S_\varepsilon^{n-1}(z) = o(1).$$

Therefore we have

$$\begin{aligned}
\frac{\partial^2 w_l}{\partial x^i \partial x^j}(x) &= \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} -\frac{\partial \Gamma}{\partial z^i}(z)\frac{\partial f_l}{\partial z^j}(x+z) + o(1) \\
&= \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \frac{\partial^2 \Gamma}{\partial z^i \partial z^j}(z) f_l(x+z) + \\
&\quad + \int_{\partial B_\varepsilon(0)} \frac{\partial \Gamma}{\partial z^i}(z)\nu^j f_l(x+z)\mathrm{d}S_\varepsilon^{n-1}(z) + o(1) \\
&= T_{ij}^\varepsilon[f_l](x) - \int_{\partial B_\varepsilon(0)} \frac{1}{\omega_n}\frac{z^i z^j}{|z|^{n+1}} f_l(x+z)\mathrm{d}S_\varepsilon^{n-1}(z) + o(1) \\
&= T_{ij}^\varepsilon[f_l](x) - \int_{\partial B_1(0)} \frac{1}{\omega_n} y^i y^j f_l(x)\mathrm{d}S^{n-1}(y) + o(1) \\
&= T_{ij}^\varepsilon[f_l](x) - \frac{\delta_{ij}}{n} f_l(x) + o(1) \quad \text{using (2.8)} \\
&=_{\lim_{\varepsilon\to 0}} T_{ij}[f_l](x) - \frac{\delta_{ij}}{n} f_l(x).
\end{aligned}$$

Thus $w_l = N[f_l]$ is a Cauchy sequence in $W^{2,p}(U)$ and $w = \lim_{l\to\infty} w_l \in W^{2,p}(U)$ with

$$\frac{\partial w}{\partial x_i} = A_1^i[f]$$

and

$$\frac{\partial^2 w}{\partial x^i \partial x^j}(x) = T_{ij}[f](x) - \frac{\delta_{ij}}{n} f(x). \tag{2.10}$$

Notice that $\sum_{i=1}^n T_{ii}^\varepsilon[f] = 0$ for all f and $\varepsilon > 0$, thus $\sum_{i=1}^n T_{ii}[f] = 0$ and we recover $-\Delta w = f$ from this formula.

The estimates also follow from Theorems 2.2.1 and 2.2.3.

□

We will see later that there is a replacement theory for these singular integral estimates in the case that $p = 1$ or $p = \infty$. More plainly we can replace L^1 by the Hardy space $\mathscr{H}^1$ and L^∞ by BMO.

There are of course counter examples in the limiting cases: Consider an example by Frehse [Fre75] $u : B_1 \subset \mathbb{R}^2 \to \mathbb{R}$ defined by $u(x, y) = u(r) = \log\log\left(\frac{e}{r}\right)$. It is easy to check that although $u \in W^{1,2}$ we do not have $u \in W^{2,1}$ even though $\Delta u \in L^1$.

Similarly we could consider $u(x, y) := xy\log\log\left(\frac{e}{r}\right)$ on B_1. We leave it to the reader to check that $\Delta u \in C^0 \subset L^\infty$ but $\nabla^2 u \notin L^\infty$.

2.4 Weak solutions and Weyl's lemma

In general we could deal with (2.1) where u and f are only distributions, in which case we say that u is a weak or distributional solution to (2.1) on $U \subset \mathbb{R}^n$ if

$$u(-\Delta\phi) = f(\phi) \tag{2.11}$$

for all $\phi \in C_c^\infty(U)$. Of course if we knew that u and f were represented by locally integrable functions then this would read

$$\int -u\Delta\phi = \int f\phi.$$

As a first regularity theorem we have the following classical result.

Theorem 2.4.1 (Weyl's Lemma)**.** *Suppose u is a distributional solution to*

$$-\Delta u = 0$$

on $U \subset \mathbb{R}^n$, then u is represented by a C^2 function and is therefore harmonic in the classical sense and analytic.

From here we may pass estimates on the Newtonian potential to other solutions:

Proposition 2.4.2. *Suppose that X and Y are normed function spaces and let $u \in L^1$ be a weak solution to*

$$-\Delta u = f \in X$$

then any estimates we have on the Newtonian potential $w := N[f]$ we can pass locally to u. I.e. if for some U' and some function space $Y(U') \subset L^1(U')$ we have

$$\|w\|_{Y(U')} \leq C\|f\|_X$$

then for any compact domain $U \subset\subset U'$ we have

$$\|u\|_{Y(U)} \leq C(\|f\|_X + \|u\|_{L^1}).$$

Proof. This follows from Weyl's lemma and standard properties of harmonic functions: We know that $u - w = h$ is harmonic and therefore $\|h\|_{L^1} \leq \|u\|_{L^1} + \|w\|_{L^1}$. Thus we have local point-wise control on h and all of its derivatives in terms of $\|u\|_{L^1} + \|w\|_{L^1}$ by standard estimates on harmonic functions. □

Clearly this is the case for $X = L^p$ and $Y = W^{2,p}$ ($p > 1$) and indeed analogous statements hold for all function spaces we consider in this thesis.

2.5 The complex theory

We briefly mention here the analogous results for the $\bar{\partial}$-operator. In particular there is a convolution operator S and kernel $\frac{1}{2\pi i z}$ such that, for $g \in C_c^\infty(\mathbb{C},\mathbb{C})$, if we set

$$\alpha(z) := S[g](z) = \frac{1}{2\pi i}\int \frac{g(\xi)}{z-\xi}\mathrm{d}\xi \wedge \mathrm{d}\bar{\xi}$$

then we have $\alpha \in C^\infty$ and $\frac{\partial \alpha}{\partial \bar{z}} = g$. Moreover there is a singular integral operator H known as the Hilbert transform or Ahlfors-Beurling transform, such that

$$\frac{\partial \alpha}{\partial z}(z) := H[g](z) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i}\int_{\mathbb{C}\backslash B_\varepsilon(z)} \frac{g(\xi)}{(z-\xi)^2}\mathrm{d}\xi \wedge \mathrm{d}\bar{\xi}.$$

These formulas will enable us to pass estimates on α and $\nabla\alpha$ given suitable estimates on the operators S and H. It should be clear that S is a Riesz potential and that H is a singular integral operator. Theorem 2.5.1 shows that H behaves like the singular integrals we have already seen (with respect to L^p estimates). In fact it is possible to show that (see [Ste70] page 60)

$$\frac{\partial \alpha}{\partial x} = -R_1(R_1 - iR_2)[g]$$

and

$$\frac{\partial \alpha}{\partial y} = -R_2(R_1 - iR_2)[g]$$

where R_i are the Riesz transforms on $\mathbb{R}^2$.

By Theorems 2.2.1 and 2.2.3 we have the following.

Theorem 2.5.1. *1. When $1 < p < \infty$, $H : L^p \to L^p$ is bounded.*

2. *When $p=1$, $H : L^1 \to L^{1,\infty}$ is bounded.*

3. *When $1<p<2$ the operator $S : L^p \to L^{\frac{2p}{2-p}}$ is bounded.*

4. *When $p=1$, $S : L^1 \to L^{2,\infty}$ is bounded.*

Remark 2.5.2. The constants $C(p)$ for which

$$\|H[g]\|_{L^p} \leq C(p)\|g\|_{L^p}$$

are known to satisfy (see [BJ08])

$$C(p) \leq \begin{cases} \frac{1.575}{p-1} & \text{if } p<2 \\ 1 & \text{if } p=2 \\ 1.575(p-1) & \text{if } p>2 \end{cases}.$$

However it is conjectured by Iwaniec [Iwa82] that one can replace 1.575 by 1.

Thus we also have the analogous results for Proposition 2.4.2 by standard estimates for holomorphic functions.

2.6 Hardy spaces and functions of bounded mean oscillation

Here we consider the Hardy space $\mathscr{H}^1$ and the local Hardy space h^1 which we think of as being replacements to L^1 for singular integral (and therefore elliptic) estimates. We will also consider the dual of $\mathscr{H}^1$, the functions of bounded mean oscillation or BMO, which are functions whose oscillation is controlled in an integral sense.

2.6.1 Maximal functions and definitions

To begin with consider the maximal function $M[f]$ defined by

$$M[f](x) := \sup_{r>0} \fint_{B_r(x)} |f(y)| \, \mathrm{d}y$$

for $f \in L^1_{loc}$, where $\fint_U = \frac{1}{|U|}\int_U$.

We first notice that if $f \in L^\infty$ then $M[f] \in L^\infty$, moreover if we had $M[f] \in L^1(\mathbb{R}^n)$ then $f \equiv 0$. The last assertion follows by considering f such that $\|f\|_{L^1(B_R)} = C > 0$ for some R (i.e. $f \not\equiv 0$). Then whenever $|x| > R$ we would have

$$M[f](x) \geq \fint_{B_{|x|+R}} |f(y)| \,\mathrm{d}y \geq \frac{C}{|x|^n}$$

so that $M[f] \notin L^1$. We will however be interested in those functions for which $M[f] \in L^1_{loc}$, which is still not the case for a general $f \in L^1$. Those functions for which $M[f]$ is locally integrable form a space called the Zygmund class or $L\log L$ (introduced in section 2.7.2) and are strongly related to the local Hardy space introduced below.

In general we have the following (due to Hardy-Littlewood in dimension one and Wiener in general), a proof of which can be found in [Zie89] or [Ste70] for example.

Theorem 2.6.1. *For $f \in L^1$, $M[f](x)$ exists and is finite almost everywhere with*

$$\lambda_{M[f]}(s) = |\{x : |M[f](x)| > s\}| \leq s^{-1} C(n) \|f\|_{L^1}.$$

If $f \in L^p$ for $1 < p \leq \infty$ then

$$\|M[f]\|_{L^p} \leq C(p,n)\|f\|_{L^p}.$$

Remark 2.6.2. Again the norm constants $C(p,n)$ can be estimated here ([Ste70, page 6]) to give

$$C(p,n) = C(n)\left(\frac{p}{p-1}\right)^{\frac{1}{p}}.$$

These estimates will be useful to us later but for now we consider another type of maximal function; one that takes account of any oscillation and cancellation properties of the function f rather than just its size. To this end pick some $\phi \in C_c^\infty(B_1)$ with $\int \phi = 1$. Let $\phi_t(z) = t^{-n}\phi\left(\frac{z}{t}\right)$ and define

$$M_\phi[f](x) := \sup_{t>0} |\phi_t * f(x)| = \sup_{t>0} \left| \int \phi_t(x-y) f(y) \,\mathrm{d}y \right|.$$

Note that we have $M_\phi[f](x) \leq CM[f](x)$ for any such ϕ. Thus we can draw the same conclusions as Theorem 2.6.1 for M_ϕ. Moreover if $f \geq 0$, $\phi \geq 0$ and $\phi \geq c$ on $B_{\frac{1}{2}}$ we also have the reverse: $M[f](x) \leq CM_\phi[f](x)$.

Definition 2.6.3. The Hardy space $\mathscr{H}^1_\phi(\mathbb{R}^n)$ is defined as all distributions f such that $M_\phi[f] \in L^1$ with norm $\|f\|_{\mathscr{H}^1_\phi} = \|M_\phi[f]\|_{L^1}$.

We see immediately that $f \in \mathscr{H}^1_\phi$ implies that f is almost everywhere equal to an L^1 function, since for such a ϕ we have $\lim_{t\downarrow 0} \phi_t * f(x) = \tilde{f}(x) \leq M_\phi[f] \in L^1$ and $\tilde{f} = f$

in the sense of distributions. We also have that $(\mathscr{H}^1_\phi, \|\cdot\|_{\mathscr{H}^1_\phi})$ is a Banach space. It also seems that we have a choice in our definition of the Hardy space – in fact this is not the case as we could pick any such ϕ and end up with an equivalent definition (and norm) – see [FS72, Theorem 11]. In which case we will suppress the ϕ appearing in the definition and write $M_\phi[f] = f_*$. Thus from now on we write $\mathscr{H}^1$ to denote the Hardy space. We also mention here that there are analogous definitions of $\mathscr{H}^p$ for all $0 < p < \infty$ – in the case that $1 < p < \infty$ we have $\mathscr{H}^p = L^p$. Moreover we reiterate here that if $f \in \mathscr{H}^1$ then we do not have $f \in L^p$ for $p > 1$ in general. Conversely if $f \in L^p$ for $p > 1$ and has compact support with $\int f = 0$, then $f \in \mathscr{H}^1$.

To get a feel for the many faces of the Hardy space we present here some equivalent definitions – our main references are the works of Fefferman-Stein [FS72], Stein [Ste70] and Semmes [Sem94].

Initially the Hardy space was considered as a subset of the holomorphic functions on the upper half plane (or disc) such that one can make distributional sense of its boundary values and that they are in L^1 (see Zygmund [Zyg02]). More precisely a holomorphic function $h : \mathbb{H} \to \mathbb{C}$ is said to be in the Hardy space H^1 if

$$\sup_{y>0} \int_{\mathbb{R}} |h(x+iy)|\, \mathrm{d}x < \infty. \tag{2.12}$$

By considering the real part of the limit of h as $y \downarrow 0$ (which exists in a suitable sense) we end up with a function $f \in \mathscr{H}^1(\mathbb{R})$! This is a truly amazing fact, and it turns out that these definitions are equivalent, i.e. one can start with a function in $\mathscr{H}^1(\mathbb{R})$ and end up with a holomorphic function $h : \mathbb{H} \to \mathbb{C}$ satisfying (2.12) and whose real boundary values are f. One uses the Poisson integral formula and the Riesz transforms to see this ([Ste70, Chapter VII, Section 3.2]), moreover the same is true in higher dimensions: Given $h = (h_0, h_1, \dots. h_n) : \mathbb{R}^{n+1}_+ \to \mathbb{R}^{n+1}$ a set of functions that satisfy the generalised Cauchy-Riemann equations such that

$$\sup_{y>0} \int_{\mathbb{R}^n} |h(x_1, x_2, \dots, x_n, y)|\, \mathrm{d}x < \infty,$$

then we say that $h \in H^1(\mathbb{R}^{n+1}_+)$. It is possible to consider the limit of h_0 as $y \downarrow 0$ and end up with a function $f \in \mathscr{H}^1(\mathbb{R}^n)$. Again we can work backwards here giving an equivalent definition of the Hardy space. Thus the Hardy spaces can be studied from either perspective in isolation; we continue down the real route, as the most important properties of the Hardy space to us is that singular integral operators are well behaved (unlike for L^1).

We also have the following properties of $\mathscr{H}^1$.

Theorem 2.6.4. *1. The singular integral operators $T_{ij}, R_i, H : \mathscr{H}^1 \to \mathscr{H}^1$ are bounded [FS72, Theorem 12].*

2. If $f \in \mathscr{H}^1$ then $\int f = 0$ see [Sem94, Proposition 1.38].

3. For all $f \in \mathscr{H}^1$ there exist $\{f_n\} \subset \mathscr{H}^1_0 = \mathscr{H}^1 \cap C_c^\infty$ such that $f_n \to f$ in $\mathscr{H}^1$ [Sem94, Proposition 1.42].

Remark 2.6.5. The most relevant result here (for us) is property 1 coupled with 3, which tells us that if $w = N[f]$ with $f \in \mathscr{H}^1$ we have $\frac{\partial^2 w}{\partial x^i \partial x^j} \in \mathscr{H}^1 \subset L^1$ with estimates. In other words we have

$$\|\nabla^2 w\|_{\mathscr{H}^1} \leq C \|f\|_{\mathscr{H}^1}.$$

An equivalent definition of $\mathscr{H}^1$ consists of those functions $f \in L^1$ such that $R_i[f] \in L^1$ for all Riesz tranforms, with norm

$$\|f\|_{\mathscr{H}^1} = \|f\|_{L^1} + \sum_{i=1}^n \|R_i[f]\|_{L^1}.$$

Thus the boundedness of such singular integrals can be thought of as given almost by definition.

We now state an important class of objects that lie in the Hardy space $\mathscr{H}^1$, discovered by Coiffman, Lions, Meyers and Semmes [CLMS93], although such examples were known to have special properties before this – see [Wen69], [Bal77] and [Mül89].

Theorem 2.6.6. *Let $E, D \in L^2(\mathbb{R}^n, \wedge^1 \mathrm{T}^* \mathbb{R}^n)$ such that*

$$\mathrm{d}^* E = 0 \quad (\mathrm{div}(E) = 0)$$

and

$$\mathrm{d} D = 0 \quad (\mathrm{curl}(D) = 0)$$

in a distributional sense. Then the inner product $\langle E, D\rangle = E \wedge *D \in \mathscr{H}^1$ with

$$\|\langle E, D\rangle\|_{\mathscr{H}^1} \leq C \|E\|_{L^2} \|D\|_{L^2}.$$

2.6.2 Functions of bounded mean oscillation – BMO

The functions of bounded mean oscillation, denoted BMO replace L^∞ in the singular integral stakes, and one might expect in keeping with this analogy that $(\mathscr{H}^1)^* = BMO$, which is indeed the case (see Theorem 2.6.9).

Since $\mathscr{H}^1 \subset L^1$ we must have $L^\infty \subset BMO$; indeed one can think of functions in BMO as being 'nearly bounded'.

Definition 2.6.7. We say that a locally integrable function $f \in BMO$ if

$$\|f\|_{BMO(\mathbb{R}^n)} := \sup_{B_r(x)\subset\mathbb{R}^n} \fint_{B_r(x)} |f(y) - f_{x,r}| \,\mathrm{d}y < \infty$$

where $f_{x,r} := \fint_{B_r(x)} f(z)\,\mathrm{d}z$.

One should think of this definition as a control on oscillation in an integral sense. The set of functions with this norm is a Banach space, except that we not only identify functions that are equal almost everywhere, but also functions that differ by a constant almost everywhere. We can also see from the definition that such a norm not only takes account of 'how big' our function is, but it really cares about the amount of oscillation also. We also have the following due to John-Nirenberg [JN61] which shows in particular that requiring $f \in BMO$ means that $f \in L^p_{loc}$ for all $p < \infty$.

Theorem 2.6.8. *There exists some $C = C(n)$ such that for all $f \in BMO(\mathbb{R}^n)$ we have*

$$\fint_{B_r(x)} e^{\left(\frac{|f(y)-f_{x,r}|}{C\|f\|_{BMO}}\right)} \mathrm{d}y \le 2.$$

Thus we really are very close to being bounded, however we have $\log|\cdot| \in BMO$ so $L^\infty \subsetneq BMO$.

The following is a famous theorem of Fefferman and Stein [FS72]:

Theorem 2.6.9. *The dual space of $\mathscr{H}^1$ is BMO; for any $F \in (\mathscr{H}^1)^*$ there exists a $g \in BMO$ such that for all $f \in \mathscr{H}^1_0$ we have*

$$F(f) = \int g(y) f(y)\,\mathrm{d}y \le C\|g\|_{BMO}\|f\|_{\mathscr{H}^1}. \tag{2.13}$$

This functional extends to the whole of $\mathscr{H}^1$ by a limiting argument. Conversely given an arbitrary $g \in BMO$, it defines a unique linear functional on $\mathscr{H}^1$ (first on $\mathscr{H}^1_0$ by integrating then by extension to the whole space) satisfying (2.13).

Remark 2.6.10. There are a couple of things worth remarking here: The first is that given an arbitrary $g \in BMO$ and $f \in \mathscr{H}^1$ the product fg is not necessarily Lebesgue integrable. Second we do not have $(BMO)^* = \mathscr{H}^1$ (again similarly with L^1 and L^∞) but there is a space $VMO (\subset BMO)$ for which we have $(VMO)^* = \mathscr{H}^1$ ([Sar75]) as per the discussion below.

We also note here that smooth functions are *not* dense in BMO with respect to this norm (as is also the case for L^∞). If this were the case then for all $f \in BMO$ we would have in particular

$$\lim_{r \downarrow 0} \fint_{B_r} |f(y) - f_{0,r}| \, \mathrm{d}y = 0$$

which is true for all smooth functions. However, considering again $\log|\cdot|$ on $\mathbb{R}^2$ we see this is not the case (recall $\log|\cdot| \in BMO$). In fact we could consider those functions for which this is the case or even consider the closure of C_c^0 with respect to the BMO norm. This is also of interest to us, and is denoted VMO (vanishing mean oscillation) which we equip with the BMO norm. We have in fact that $(VMO)^* = \mathscr{H}^1$ due to Sarason [Sar75] – see also [Daf02]. We denote $VMO_0 = C_c^0 \cap BMO$.

Theorem 2.6.11. *$(VMO)^* = \mathscr{H}^1$, so that given any operator $F \in (VMO)^*$ then there exists some $f \in \mathscr{H}^1$ such that for all $g \in VMO_0$ we have*

$$F(g) = \int f(y) g(y) \, \mathrm{d}y \leq C \|f\|_{\mathscr{H}^1} \|g\|_{BMO}$$

and we may extend this to all of VMO by approximation. Conversely any $f \in \mathscr{H}^1$ defines a linear functional on VMO.

This theorem also yields a 'weak star' convergence property for $\mathscr{H}^1$. Thus given a bounded sequence $\{f_n\} \subset \mathscr{H}^1$ we could find a subsequence such that $f_n \stackrel{*}{\rightharpoonup} f \in \mathscr{H}^1$.

The space BMO is also well behaved with respect to singular integral operators, in particular we have

Theorem 2.6.12. *$R_i, T_{ij}, H : BMO \to BMO$ are bounded.*

This theorem also first appeared in [FS72], and was one of the main ingredients in the proof of Theorem 2.6.9. In section 2.8 we shall see a generalisation of Theorem 2.6.12 (due to Jaak Peetre [Pee66]) for Campanato spaces.

2.6.3 Local Hardy space, bmo and vmo

Now we introduce the local Hardy space h^1 introduced by Goldberg [Gol79]. This function space captures the essence of $\mathscr{H}^1$ without it being so particular with respect to multiplication of other functions (remember $f \in \mathscr{H}^1$ implies $\int f = 0$ so $\mathscr{H}^1$ is not stable even up to multiplication of smooth functions!). In particular one can use cut-off functions in dealing with $f \in h^1$ which is invaluable in PDE theory. Moreover we can make sense of h^1 on manifolds although we do not need this fact here.

For a distribution f as before we consider a maximal function with respect to some smooth $\phi \in C_c^\infty(B_1)$ with $\int \phi = 1$. To this end we define

$$\tilde{M}_\phi[f](x) := \sup_{0<t<1} |\phi_t * f(x)|.$$

Suppose also that f is supported on some domain U, then we consider

$$\tilde{\tilde{M}}_\phi[f](x) := \sup_{0<t<\min\{1,\mathrm{dist}(x,\partial U)\}} |\phi_t * f(x)|.$$

Definition 2.6.13. We say that $f \in h^1(\mathbb{R}^n)$ if $\tilde{M}_\phi[f] \in L^1$ with norm $\|f\|_{h^1} = \|\tilde{M}_\phi[f]\|_{L^1}$.
We say that $f \in h^1(U)$ if $\tilde{\tilde{M}}_\phi[f] \in L^1(U)$ with norm $\|f\|_{h^1(U)} = \|\tilde{\tilde{M}}_\phi[f]\|_{L^1(U)}$.

Again this definition is independent of such ϕ up to equivalence of norms. Clearly (as for $\mathscr{H}^1$) we have $h^1 \subset L^1$, moreover $\mathscr{H}^1 \subset h^1$. One can also come at the local Hardy space $h^1(\mathbb{R})$ (as in the case for $\mathscr{H}^1$) by considering boundary values of certain holomorphic functions on the 'upper half strip' $S = \{(x,y) \in \mathbb{H} : 0 < y < 1\}$ and analogously in higher dimensions – see [Gol79].

Proposition 2.6.14. *Let $f \in h^1(\mathbb{R}^n)$*

1. *for $\psi \in C_c^\infty$ with $\int \psi \neq 0$ there exists a constant λ such that $\psi(f-\lambda) \in \mathscr{H}^1$ with*

$$\|\psi(f-\lambda)\|_{\mathscr{H}^1} \leq C\|f\|_{h^1}$$

and $C = C(\psi)$ (λ is chosen so that $\int \psi(f-\lambda) = 0$)

2. *moreover if $\phi \in C^{0,\alpha}$ then $\phi f \in h^1$ and*

$$\|\phi f\|_{h^1} \leq C(\alpha)\|\phi\|_{C^{0,\alpha}}\|f\|_{h^1}.$$

Remark 2.6.15. We can conclude that if $w = N[f]$ with $f \in h^1$, then $\nabla^2 w \in L^1_{loc}$ with an estimate. In particular there is a $C = C(U,n)$ such that for bounded $U \subset \mathbb{R}^n$ we have

$$\|\nabla^2 w\|_{L^1(U)} \leq C\|f\|_{h^1}.$$

As for the global Hardy space we have a related space of nearly bounded functions bmo and the subspace $vmo \subset bmo$ which is the closure with respect to the bmo norm of smooth compactly supported functions, with $(h^1)^* = bmo$ and $(vmo)^* = h^1$ (see [Gol79] and [Daf02] for details).

2.7 Rearrangement invariant spaces

These are spaces of functions for which only their size matters, in that we are free to rearrange our function to a more symmetric shape (radial and non increasing in r) and this rearrangement does not affect the properties we wish to measure. Thus we do not take into account any cancellation properties of our function and we measure it in a sense that only cares 'on how big a set our function is big' rather than 'where it is big'. It is thus unsurprising that the key object we use is

$$\lambda_f(s) = |\{x : |f(x)| > s\}|.$$

Roughly speaking the radial non-increasing rearrangement associated to a function f, denoted f_r^* is obtained by imagining (let us say $f \in L^1$ is defined on $\mathbb{R}^n$) the graph of $|f|$ over $\mathbb{R}^n$ is made out of a sheet of rubber. We then wish to move this sheet so that the volume underneath remains unchanged whilst it becomes radially symmetric and non-increasing in r. This procedure of course requires us to push any singularities to the origin and all the places where f is small get pushed out to ∞ and we are left with something that (depending on the smoothness of f) looks a bit like a big top. Now this function f_r^* contains all the information about the integrability of $|f|$, in particular

$$\lambda_{f_r^*}(s) = \lambda_f(s)$$

and this defines f_r^* uniquely in such a class (i.e. there is only one such rearrangement).

We could now simplify $|f|$ even further by taking the one dimensional analogue of f_r^* (take re-scaled radial slice) by defining

$$f^*(t) = f_r^*\left(\left(\frac{t}{n\omega_n}\right)^{\frac{1}{n}}\right)$$

so that

$$\begin{aligned}
\lambda_{f^*}(s) &= |\{t \in \mathbb{R} : f^*(t) > s\}| \\
&= \sup\{t \in \mathbb{R} : f^*(t) > s\} \\
&= \sup\left\{t \in \mathbb{R} : f_r^*\left(\left(\frac{t}{n\omega_n}\right)^{\frac{1}{n}}\right) > s\right\} \\
&= \sup\{n\omega_n t^n : f_r^*(t) > s\} \\
&= |\{x \in \mathbb{R}^n : f_r^*(x) > s\}| \\
&= \lambda_f(s). \qquad (2.14)
\end{aligned}$$

We could have simply started with the definition

$$f^*(t) := \inf\{s \geq 0 : \lambda_f(s) \leq t\},$$

which satisfies (2.14), is non increasing and is virtually the inverse of λ_f (see [Zie89] for details). By definition we also have

$$f^*(\lambda_f(s)) \leq s.$$

We wish to see now in what way f^* captures integrability of f. To this end consider a smooth strictly increasing function $\phi : (0,\infty) \to (0,\infty)$ with $\phi(0) = 0$. Later we will require that ϕ be at least convex, but for now we do not need this extra condition. Now suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is measurable and that $\int \phi(|f(x)|)\,\mathrm{d}x < \infty$ then using Fubini's theorem we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \phi(|f(x)|)\,\mathrm{d}x &= \int_{\mathbb{R}^n}\int_0^\infty \chi_{\{(x,s):\phi(|f(x)|)>s\}}\,\mathrm{d}s\mathrm{d}x \\
&= \int_0^\infty \int_{\mathbb{R}^n} \chi_{\{(x,s):\phi(|f(x)|)>s\}}\,\mathrm{d}x\mathrm{d}s \\
&= \int_0^\infty \lambda_f(\phi^{-1}(s))\,\mathrm{d}s \\
&= \int_0^\infty \lambda_{f^*}(z)\phi\prime(z)\,\mathrm{d}z \\
&= \int_0^\infty \phi(f^*(t))\,\mathrm{d}t. \qquad (2.15)
\end{aligned}$$

Where we have also used heavily the properties of ϕ, with $s = \phi(z)$ and also for instance $\phi(|f(x)|) > s \iff |f(x)| > \phi^{-1}(s)$ since ϕ is strictly increasing and therefore has a strictly increasing inverse.

Examples of rearrangement invariant function spaces are well known to us, for instance the standard Lebesgue spaces L^p could be characterised by either $|f|^p \in L^1$ or $(f^*)^p \in L^1$, you can see this by letting $\phi(z) = z^p$.

2.7.1 Lorentz spaces

As already mentioned, the information on where f is big is all contained at 0 for f^*, moreover the information on where f is small is contained at ∞ for f^*. The point of Lorentz spaces is to interpolate between Lebesgue spaces – which provide a rather coarse distinction between functions in comparison. Indeed we could have introduced Lorentz spaces in a pure interpolation sense, at which point many interpolation type results follow much more readily, see [Tar07] for an introduction from this perspective.

We feel that using non-increasing rearrangements provide a more intuitive feel as to what we are measuring. To that end we write down some quantities for $1 < p < \infty$ and $1 \leq q \leq \infty$:

$$|f|_{L^{p,q}} := \begin{cases} \left(\int_0^\infty t^{\frac{q}{p}-1}(f^*(t))^q \,\mathrm{d}t\right)^{\frac{1}{q}} & \text{if } q < \infty \\ \sup_{t>0} t^{\frac{1}{p}} f^*(t) & \text{if } q = \infty. \end{cases} \tag{2.16}$$

When $p = 1$ we also consider weak L^1, i.e.

$$|f|_{L^{1,\infty}} := \sup_{t>0} t f^*(t)$$

and we have no analogue for L^∞. The reader can check that the spaces $L^{p,\infty}$ are equivalent to the ones introduced earlier in section 2.2.1.

We define the Lorentz spaces $L^{p,q} := \{f : |f|_{L^{p,q}} < \infty\}$. First off we notice that $L^{p,p} = L^p$, and when $q < p$ we are making the singularity of f^* at the origin slightly worse (although we are integrating against a lower power of f^*) and tempering the decay at ∞. The converse statement holds when $q > p$. We won't go into details here, however we have the following inclusions for $1 < q_1 < p < q_2 < \infty$ – see [Zie89] for details.

$$L^{p,1} \subset L^{p,q_1} \subset L^{p,p} = L^p \subset L^{p,q_2} \subset L^{p,\infty}.$$

On a domain $U \subset \mathbb{R}^n$ with $|U| < \infty$, and $p_1 < p < p_2$ we have

$$L^{p_2,\infty}(U) \subset L^{p,1}(U) \subset L^{p,\infty}(U) \subset L^{p_1,1}(U).$$

As suggested by the notation the quantities $|\cdot|_{L^{p,q}}$ are not norms, however if we replace f^* by

$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s$$

then the corresponding quantities $\|f\|_{L^{p,q}}$ (formed by replacing f^* by f^{**} in (2.16)) are norms and we have

$$\frac{1}{C_{p,q}}|f|_{L^{p,q}} \leq \|f\|_{L^{p,q}} \leq C_{p,q}|f|_{L^{p,q}}$$

(see [Hun66]).

The Lorentz spaces lend themselves to fine tuning L^p theory, for instance one can extend the bounds attained for the Riesz potentials (and therefore Sobolev embeddings) in Proposition 2.2.1 (see [Zie89, Theorem 2.10.2] for a direct proof or use an interpolation theorem [Hun66]), although we do not explicitly state the extension here. We also have the following which follows from interpolation [Hun66]

Theorem 2.7.1. *$R_i, T_{ij}, H : L^{p,q} \to L^{p,q}$ are bounded for $1 < p < \infty$ and $1 \le q \le \infty$.*

Some other important things to mention are that smooth functions are dense in $L^{p,q}$ if $1 < p < \infty$ and $1 \le q < \infty$ but they are not dense in $L^{p,\infty}$. Moreover $(L^{p,q})^* = L^{\frac{p}{p-1},\frac{q}{q-1}}$ when $1 < q < \infty$; and $(L^{p,1})^* = L^{\frac{p}{p-1},\infty}$ but not vice versa. (These facts are analogous as with L^1 and L^∞ – see [Hun66].)

The following is an example of an improved embedding one has when considering Lorentz spaces (see [Hél02, Theorem 3.3.10] for a proof). We note that the proof in [Hél02] does not follow from Riesz potential estimates, but uses the Pólya-Szegő inequality (see [BZ88]).

Theorem 2.7.2. *We have the continuous embedding $W^{1,1}(\mathbb{R}^n) \hookrightarrow L^{\frac{n}{n-1},1}(\mathbb{R}^n)$ when $n \ge 2$.*

We also have the following miscellaneous embeddings and estimates.

Lemma 2.7.3. 1. *The embedding $W^{n-1,(\frac{n}{n-1},1)}(\mathbb{R}^n) \hookrightarrow C^0$ is continuous.*

2. *The embedding $W^{1,(n,\infty)}(\mathbb{R}^n) \hookrightarrow BMO(\mathbb{R}^n)$ is continuous.*

3. *The embedding $W^{n,1}(\mathbb{R}^n) \hookrightarrow C^0$ is continuous.*

One can prove part 1 by using Remark 2.2.2 and noticing that for the Riesz potential A_{n-1} the kernel lies in $L^{n,\infty}$. Thus the duality between $L^{\frac{n}{n-1},1}$ and $L^{n,\infty}$ can be used with an approximation argument to conclude the result (recall smooth functions are dense in $L^{\frac{n}{n-1},1}$).

Part 2 follows by applying the Poincaré inequality and a scaling argument (see appendix B). For $f \in W^{1,(n,\infty)}$ we have

$$\|f - f_{x,r}\|_{L^1(B_r(x)} \le Cr\|\nabla f\|_{L^1(B_r(x))} \le Cr^n\|\nabla f\|_{L^{n,\infty}(B_r(x))}.$$

Part 1 coupled with Theorem 2.7.2 proves part 3. However this also follows by writing

$$f(x) = \int_{-\infty}^{x^1} \dots \int_{-\infty}^{x_n} \frac{\partial^n f}{\partial x_1 \dots \partial x_n}(y_1, \dots, y_n)\, \mathrm{d}y$$

which holds for all $f \in C_c^\infty$.

Remark 2.7.4. We remark here a useful consequence of part 1 of Lemma 2.7.3. We know that $N : h^1(\mathbb{R}^2) \to W^{2,1}(U)$ is bounded for all bounded domains $U \subset \mathbb{R}^2$. (Since $\mathscr{H}^1 \subset h^1$ we have the same for $\mathscr{H}^1$.) As a consequence of this $\nabla N : h^1(\mathbb{R}^2) \to L^{2,1}(U)$ is bounded.

2.7.2 Orlicz spaces

Now we wish to look at spaces which are defined by some Orlicz function $\phi : [0,\infty) \to [0,\infty)$ in the sense that $f \in L^{\phi}$ if $\int \phi(\mu|f|) < \infty$ for some $\mu > 0$. We say that ϕ is an Orlicz function if it is smooth and increasing, $\phi(0) = 0$ and $\phi(t) \to \infty$ as $t \to \infty$. In which case we introduce the following quantity, known as the Luxembourg functional.

$$|f|_{L^{\phi}} := \inf\left\{\lambda > 0 : \int \phi\left(\frac{|f|}{\lambda}\right) \leq \phi(1)\right\}.$$

We call an Orlicz function a Young function if it is also convex. In this case we know that $(L^{\phi}, |\cdot|_{L^{\phi}})$ is a Banach space (in particular $|\cdot|_{L^{\phi}}$ is a norm-see [RR91, Theorem 3]). An important class of spaces is the Zygmund class $L\log^{\frac{1}{\alpha}} L$ and their duals Exp^{α} generated by Young functions $t(\log(e+t))^{\frac{1}{\alpha}}$ and $e^{t^{\alpha}} - 1$ respectively.

The class $L\log^{\beta} L$ is a Banach space for all $\beta \geq 0$ with a norm also given by

$$\|f\|_{L\log^{\beta} L} = \int |f(x)| \left(\log\left(e + \frac{|f(x)|}{\|f\|_{L^1}}\right)\right)^{\beta} \mathrm{d}x = \int f^*(t) \left(\log\left(e + \frac{f^*(t)}{\|f\|_{L^1}}\right)\right)^{\beta} \mathrm{d}t$$

by (2.15). The fact that this is a norm is non-trivial (the only non-trivial thing to check is the triangle inequality) a proof of which can be found in [IV99]. For now though one can check that the following quantity

$$|f|_{L\log^{\beta} L} = \int f^*(t) \left(\log\left(e + \frac{1}{t}\right)\right)^{\beta} \mathrm{d}t$$

is equaivalent.

Proof. Suppose $|f|_{L\log^{\beta} L} < \infty$, then clearly $f \in L^1$ (with $\|f\|_{L^1} \leq |f|_{L\log L^{\beta} L}$), which implies

$$f^*(t) \leq \frac{\|f\|_{L^1}}{t}$$

and therefore

$$\|f\|_{L\log^{\beta} L} \leq |f|_{L\log^{\beta} L}.$$

For the reverse inclusion suppose $f \in L\log^{\beta} L$, then again $f \in L^1$ with $\|f\|_{L^1} \leq \|f\|_{L\log^{\beta} L}$. Setting

$$X = [0,1] \cap \left\{t : f^*(t) \geq \frac{\|f\|_{L^1}}{t^{\frac{1}{2}}}\right\} \subset [0,\infty),$$

and $Y = X^c$ we have

$$
\begin{aligned}
|f|_{L\log^\beta L} &= \int_0^\infty f^*(t)\left(\log\left(e+\frac{1}{t}\right)\right)^\beta \\
&\le \int_X f^*(t)\left(\log\left(\frac{4}{t}\right)\right)^\beta + \int_Y f^*(t)\left(\log\left(e+\frac{1}{t}\right)\right)^\beta \\
&\le C\left(\|f\|_{L\log^\beta L} + \|f\|_{L^1}\left(\int_0^1 t^{-\frac{1}{2}}\left(\log\left(e+\frac{1}{t}\right)\right)^\beta \mathrm{d}t + 1\right)\right) \\
&\le C\|f\|_{L\log^\beta L}.
\end{aligned}
$$

□

Thus we can define $L\log^\beta L := \{f : |f|_{L\log^\beta L} < \infty\}$ with norm given by $\|f\|_{L\log^\beta L}$. We have the following fundamental property (see [Ste69] for a proof).

Lemma 2.7.5. *$f \in L\log L$ if and only if $M[f] \in L^1_{loc}$. In particular if f is supported on some bounded domain U, then the quantities $\|M[f]\|_{L^1(U)}$ and $\|f\|_{L\log L(U)}$ are equivalent.*

Remark 2.7.6. An important remark here is that if $f \in L\log L(U)$ for a bounded domain U, then $f \in h^1(\mathbb{R}^n)$ (upon extending by zero) with a trivial estimate. This follows from a comparison of the functions $\tilde{M}_\phi[f]$ and $M[f]$, along with Lemma 2.7.5. Thus we also have

$$(f - \fint_U f)\chi_U \in \mathscr{H}^1$$

by Proposition 2.6.14.

The converse holds true in the following sense: If $f \in h^1(\mathbb{R}^n)$ and $f \ge 0$ then $f \in L\log L_{loc}$ – again this follows by a comparison of the functions $\tilde{M}_\phi[f]$ and $M[f]$, coupled with the fact that we are free to pick ϕ such that $\phi \ge 0$ and $\phi \ge c$ on $B_{\frac{1}{2}}$. In this case we have:

$$M[f] \le C(\tilde{M}_\phi[f] + \|f\|_{L^1}).$$

We also have (see [IM01] and [RR91])

Lemma 2.7.7. *For all $0 \le \beta \le 1$ we have $(L\log^\beta L)^* = Exp^{\frac{1}{\beta}}$.*

This result follows from a more general result about Orlicz spaces generated by Young functions [IM01, Theorem 4.12.1]. Just like in the case of L^1 and L^∞ we do not have the converse. We also remark that when $\beta > 1$, $e^{\frac{1}{\beta}} - 1$ is not a Young function.

2.7.3 Critical Sobolev embedding

In the critical exponent Sobolev embedding with the Rellich-Kondrachov compactness theorem we have $W_0^{1,n}(U) \subset\subset L^p(U)$ for all finite p and bounded domains $U \subset \mathbb{R}^n$. A more detailed ananlysis of the Riesz potential estimates on bounded domains yields the following result due to Trudinger [Tru67], see also [Zie89].

Theorem 2.7.8. *Given a bounded domain $U \subset \mathbb{R}^n$ we have the continuous embedding*

$$W_0^{1,n}(U) \hookrightarrow Exp^{\frac{n}{n-1}}(U).$$

Recall from the definition of the Exp^α norms that this is equivalent to saying that for all $f \in W_0^{1,n}(U)$ there is a C such that

$$\int_U (e^{\left[\left(\frac{|f|}{C\|\nabla f\|_{L^n(U)}}\right)^{\frac{n}{n-1}}\right]} - 1) \leq e - 1$$

or in other words that

$$\int_U e^{\left[\left(\frac{|f|}{C\|\nabla f\|_{L^n(U)}}\right)^{\frac{n}{n-1}}\right]} \leq e - 1 + |U|.$$

Using theorem 2.7.8 and the Rellich-Kondrachov compactness theorem we will prove the following (we expect this result is well known within certain communities).

Theorem 2.7.9. *Given a bounded domain $U \subset \mathbb{R}^n$ the following embedding*

$$W_0^{1,n}(U) \subset\subset Exp^\alpha(U)$$

is compact for all $1 \leq \alpha < \frac{n}{n-1}$.

Proof. To begin with let $\{f_k\} \subset W_0^{1,n}(U)$ be a bounded sequence. By standard theorems (Rellich-Kondrachov, Banach-Alaouglu) we can choose a subsequence (not re-labelled) and $f \in W_0^{1,n}$ satisfying the following:

- $f_k \rightharpoonup f$ in $W_0^{1,n}(U)$,
- $f_k \overset{*}{\rightharpoonup} f$ in $Exp^{\frac{n}{n-1}}(U)$,
- $f_k \to f$ in $L^2(U)$.

Indeed we can replace the sequence $\{f_k\}$ by $\{f_k - f\}$ and (again without re-labelling) we can put ourselves in the position where

- $f_k \rightharpoonup 0$ in $W_0^{1,n}(U)$,

- $f_k \overset{*}{\rightharpoonup} 0$ in $Exp^{\hat{\alpha}}(U)$ for all $1 \leq \hat{\alpha} \leq \frac{n}{n-1}$,
- $f_k \to 0$ in $L^2(U)$.

Thus our goal is to prove that $f_k \to 0$ in Exp^{α} for any $1 \leq \alpha < \frac{n}{n-1}$.

To that end we must check that

$$\|f_k\|_{Exp^{\alpha}(U)} = \inf\left\{\lambda : \int_U e^{\left(\frac{|f_k|}{\lambda}\right)^{\alpha}} \leq e - 1 + |U|\right\} \to_{k\to\infty} 0.$$

For reasons that will become apparent pick arbitrary $0 < \varepsilon < \log\left(1 + \frac{1}{|U|}\right)$ and write

$$\left(\frac{|f_k|}{\varepsilon}\right)^{\alpha} = \left(\frac{\left(\frac{2\alpha(n-1)}{n}\right)^{\frac{n-1}{n}} \|f_k\|_{Exp^{\frac{n}{n-1}}}}{\varepsilon}\right)^{\alpha} \left(\frac{|f_k|}{\left(\frac{2\alpha(n-1)}{n}\right)^{\frac{n-1}{n}} \|f_k\|_{Exp^{\frac{n}{n-1}}}}\right)^{\alpha}$$

and let

$$a = \left(\frac{\left(\frac{2\alpha(n-1)}{n}\right)^{\frac{n-1}{n}} \|f_k\|_{Exp^{\frac{n}{n-1}}}}{\varepsilon}\right)^{\alpha}$$

and

$$b = \left(\frac{|f_k|}{\left(\frac{2\alpha(n-1)}{n}\right)^{\frac{n-1}{n}} \|f_k\|_{Exp^{\frac{n}{n-1}}}}\right)^{\alpha}.$$

Setting $q := \frac{n}{\alpha(n-1)} > 1$ and definig p by $\frac{1}{p} + \frac{1}{q} = 1$ we have by Young's inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

To cut a short story even shorter we end up with

$$\begin{aligned}
\left(\frac{|f_n|}{\varepsilon}\right)^{\alpha} &\leq \frac{n-\alpha(n-1)}{n}\left(\frac{\left(\frac{2\alpha(n-1)}{n}\right)^{\frac{n-1}{n}} \|f_k\|_{Exp^{\frac{n}{n-1}}}}{\varepsilon}\right)^{\frac{\alpha n}{n-\alpha(n-1)}} \\
&+ \frac{1}{2}\left(\frac{|f_k|}{\|f_k\|_{Exp^{\frac{n}{n-1}}}}\right)^{\frac{n}{n-1}} \\
&\leq K_1(n,\alpha,\varepsilon) + \frac{1}{2}\left(\frac{|f_k|}{\|f_k\|_{Exp^{\frac{n}{n-1}}}}\right)^{\frac{n}{n-1}},
\end{aligned} \tag{2.17}$$

where we have used crucially that $\alpha < \frac{n}{n-1}$ and that $\|f_k\|_{Exp^{\frac{n}{n-1}}}$ is uniformly bounded.

Now let $K_2(\varepsilon,\alpha) = \varepsilon^{1+\frac{1}{\alpha}}$ so that whenever $|f_k| \geq K_2$ (2.17) gives us that

$$\left(\frac{|f_k|}{\varepsilon}\right)^{\alpha} \leq K_1 \log\left(\frac{|f_k| e}{K_2}\right) + \frac{1}{2}\left(\frac{|f_k|}{\|f_k\|_{Exp^{\frac{n}{n-1}}}}\right)^{\frac{n}{n-1}}$$

thus

$$\begin{aligned}
\int_{|f_k|\geq K_2} e^{\left(\frac{|f_k|}{\varepsilon}\right)^{\alpha}} &\leq K_3(n,\varepsilon,\alpha)\int_U |f_k| e^{\frac{1}{2}\left(\frac{|f_k|}{\|f_k\|_{Exp^{\frac{n}{n-1}}}}\right)^{\frac{n}{n-1}}} \\
&\leq K_3 \|f_k\|_{L^2(U)} \left(\int_U e^{\left(\frac{|f_k|}{\|f_k\|_{Exp^{\frac{n}{n-1}}}}\right)^{\frac{n}{n-1}}}\right)^{\frac{1}{2}} \\
&\leq K_3 \|f_k\|_{L^2(U)} (|U| + e - 1)^{\frac{1}{2}}.
\end{aligned}$$

Using this we have

$$\begin{aligned}
\int_U e^{\left(\frac{|f_k|}{\varepsilon}\right)^{\alpha}} &= \int_{|f_k|<K_2} e^{\left(\frac{|f_k|}{\varepsilon}\right)^{\alpha}} + \int_{|f_k|\geq K_2} e^{\left(\frac{|f_k|}{\varepsilon}\right)^{\alpha}} \\
&< |U| e^{\varepsilon} + K_3 \|f_k\|_{L^2(U)} (|U| + e - 1)^{\frac{1}{2}} \leq |U| + e - 1
\end{aligned}$$

when k is large enough. Therefore we have shown that for all $0 < \varepsilon < \log\left(1 + \frac{1}{|U|}\right)$ we have

$$\|f_k\|_{Exp^{\alpha}(U)} \leq \varepsilon$$

for sufficiently large k and we are done. □

2.8 Morrey and Campanato spaces

These are also variations on L^p spaces, however they are not rearrangement invariant, nor can they be thought of as interpolation spaces for L^p, and in general, smooth functions are not dense in these spaces. However they turn out to be well behaved with respect to Riesz potential and singular integral estimates.

The starting point for a function belonging to such a space is that you are locally in L^p for some p, but further to this, the L^p norm on small balls must decay at a fast enough rate as the size of the balls decreases. Our main reference here for the definitions and basic results is [Gia83, Chapter III].

First we introduce the Morrey spaces $M^{p,\beta}(E)$ for $1 \leq p < \infty$ and $0 \leq \beta \leq n$ ($E \subset \mathbb{R}^n$). We say that $g \in M^{p,\beta}(E)$ if

$$M_\beta[g^p](x) := \sup_{r>0} r^{-\beta} \int_{B_r(x)\cap E} |g|^p \in L^\infty$$

with norm (which makes $M^{p,\beta}$ a Banach space)

$$\|g\|_{M^{p,\beta}(E)} = \|M_\beta[g^p]^{\frac{1}{p}}\|_{L^\infty(E)}.$$

Clearly we have $M^{p,0} = L^p$ and $M^{p,n} = L^\infty$, also we see that $M_n[\cdot] = M[\cdot]$ is the usual maximal function up to a constant. Also note that if we allow $\beta > n$ then $M^{p,\beta} = \{0\}$. We also mention the obvious inclusions given by Hölder's inquality; that $M^{p,\beta} \hookrightarrow M^{q,\hat{\beta}}$ when $q \leq p$ and $n - \hat{\beta} = \frac{q}{p}(n-\beta)$.

The related Campanato spaces $\mathscr{C}^{p,\beta}$ are variations on BMO, thus we try to capture an integral measure of oscillation similar to that for the Morrey spaces. For $g \in L^1(E)$ let $g_{r,x} = \fint_{B_r(x)\cap E} g$ and we say that $g \in \mathscr{C}^{p,\beta}(E)$ if $g \in L^p(E)$ and

$$[g]_{\mathscr{C}^{p,\beta}(E)} := \sup_{x\in E,\, r>0} \left(r^{-\beta} \int_{B_r(x)\cap E} |g - g_{r,x}|^p \right)^{\frac{1}{p}} < \infty$$

with norm (making $\mathscr{C}^{p,\beta}$ Banach spaces)

$$\|g\|_{\mathscr{C}^{p,\beta}(E)} = [g]_{\mathscr{C}^{p,\beta}(E)} + \|g\|_{L^p(E)}.$$

For Lipschitz domains we have $M^{p,\beta} = \mathscr{C}^{p,\beta}$, when $0 \leq \beta < n$ [Gia83, Chapter III, Proposition 1.2]. However (modulo constants) $\mathscr{C}^{p,n} = BMO \cap L^p$ for all p as opposed to $M^{p,n} = L^\infty$. We actually have that $M^{p,\beta} \subset \mathscr{C}^{p,\beta}$ with a uniform estimate (in n, p and β). The reverse inclusion holds with an estimate whose constant blows up as β approaches n.

Moreover $\mathscr{C}^{p,\beta}$ makes sense for $\beta > n$ and when $n < \beta \leq n + p$ we have $\mathscr{C}^{p,\beta} = C^{0,\gamma}$ with $\gamma = \frac{\beta - n}{p}$ [Gia83, Chapter III, Theorem 1.2]. If $\beta > n + p$ then $\mathscr{C}^{p,\beta}$ are the constant functions.

We say that $g \in M_k^{p,\beta}$ if $g, \nabla^k g \in M^{p,\beta}$. Using the Poincaré inequality we see that if $g \in M_1^{p,\beta}$ for some $0 \leq \beta \leq n$ then $g \in \mathscr{C}^{p,p+\beta}$. Therefore if $n - p < \beta \leq n$ then $g \in C^{0,\frac{\beta+p-n}{p}}$. Also the borderline case ($\beta = n - p$) gives $g \in BMO$. These last facts yield another proof of the Morrey embedding theorem: suppose $g \in W^{1,q}$ for $q > n$. Then $g \in M_1^{n,n-\frac{n^2}{q}} \hookrightarrow C^{0,1-\frac{n}{q}}$.

We also introduce here the related weak Morrey spaces $M^{(p,\infty),\beta}(E)$, consisting of functions g in the Lorentz space $L^{(p,\infty)}(E)$ or 'weak L^p' with

$$\|g\|_{M^{(p,\infty),\beta}(E)} := \sup_{x,\, r>0} r^{-\frac{\beta}{p}} \|g\|_{L^{(p,\infty)}(B_r(x)\cap E)} < \infty.$$

This condition is equivalent to

$$|\{x \in B_r(x_0) \cap E : |g|(x) > s\}| \le C\|g\|^p_{M^{(p,\infty),\beta}(E)} s^{-p} r^{\beta}$$

with C independent of x_0 and r.

2.8.1 Singular integrals and Riesz potentials on Morrey-Campanato spaces

Even though the Campanato spaces do not interpolate the L^p spaces we still have good estimates on singular integrals. We also have the following result of Peetre [Pee66] which generalises both Calderon-Zygmund and Schauder estimates.

Theorem 2.8.1. *$T_{ij}, R_i, H : \mathscr{C}^{p,\beta} \to \mathscr{C}^{p,\beta}$ are bounded for $\beta < n + p$.*

Therefore we also have for $1 < p < \infty$ and $0 \le \beta < n$ that $\nabla^2 N : M^{p,\beta} \to M^{p,\beta}$ is bounded.

We will also show here that we have the following extension of the Riesz potential estimates, due to Adams [Ada75]. A few words of warning are necessary here: Our notation differs from that used in [Ada75], in particular we define our maximal functions M_β differently and the Riez potentials we consider are slightly more general. We provide proofs for the following because we need a slight extension of one of the results, moreover we wish to keep track of some constants a little more closely.

Theorem 2.8.2. *Let $0 \le \beta < n$, $0 < \alpha p < n - \beta$ and $\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{\alpha}{n-\beta}$ (recall $0 < \alpha < n$).*

1. *When $p > 1$ we have that*
$$A_\alpha : M^{p,\beta} \to M^{\tilde{p},\beta}$$
is bounded. Moreover there exists
$$C(n,p,\alpha,\beta) \le C(n)\left(\frac{p}{p-1}\right)^{\frac{1}{\tilde{p}}} \sup\left\{\frac{1}{1-\left(\frac{1}{2}\right)^{\alpha}}, \frac{1}{1-\left(\frac{1}{2}\right)^{\alpha-\frac{n-\beta}{p}}}\right\}$$
such that
$$\|A_\alpha[g]\|_{M^{\tilde{p},\beta}} \le C(n,p,\alpha,\beta)\|g\|_{M^{p,\beta}}.$$

2. *When* $p = 1$ *we have that*

$$A_\alpha : M^{1,\beta} \to M^{(\tilde{p},\infty),\beta}$$

is bounded. Moreover there exists

$$C(n,\alpha,\beta) \le C(n) \sup \left\{ \frac{1}{1-\left(\frac{1}{2}\right)^{\alpha}}, \frac{1}{1-\left(\frac{1}{2}\right)^{\alpha-(n-\beta)}} \right\}$$

such that

$$\|A_\alpha[g]\|_{M^{(\tilde{p},\infty),\beta}} \le C(n,\alpha,\beta)\|g\|_{M^{p,\beta}}.$$

These reduce to well known estimates when $\beta = 0$ (Theorem 2.2.1).

Remark 2.8.3. Setting $\alpha = 1$ and recalling that $\nabla N[f] = \nabla w$ is an operator of the form A_1 (see (2.9)) we see that given a, b with $1 < a \le p \le b < n - \beta$ then there is a uniform constant $C = C(a, b)$ such that for $g \in M^{p,\beta}$ then

$$\|\nabla N[g]\|_{M^{\tilde{p},\beta}} \le C\|g\|_{M^{p,\beta}}$$

for any p and β in this range.

Given Remark 2.2.2 we also see that these estimates extend the Sobolev embedding theorems for $g \in M_1^{p,\beta}$ and $p < n - \beta$.

In order to prove Theorem 2.8.2 we present a small extension of a result of Adams [Ada75] giving improved estimates on Riesz potentials acting on functions in some appropriate Morrey space. We state the general theory here as we have not seen the proof of Proposition 2.8.4 elsewhere, although it is really an amalgamation of the proofs of Theorem 1.77 in [Sem94], and Proposition 3.1 in [Ada75].

This proposition is a replacement of a weak L^q-estimate given by Proposition 3.2 in [Ada75]. We replace the borderline case $p = 1$ by the Hardy space, thereby giving the strong estimate (we replace $M[g]$ by g_*).

Proposition 2.8.4. *Let* $0 \le \beta < n$ *and* $0 < \alpha < n - \beta$, *then if* $g \in M^{1,\beta}$

$$|A_\alpha[g]| \le C(\alpha,\beta,n)(M_\beta[g](x))^{\frac{\alpha}{n-\beta}}(g_*(x))^{\frac{n-\beta-\alpha}{n-\beta}}$$

where

$$C(\alpha,\beta,n) \le C(n) \sup \left\{ \frac{1}{1-\left(\frac{1}{2}\right)^{\alpha}}, \frac{1}{1-\left(\frac{1}{2}\right)^{\alpha-(n-\beta)}} \right\}.$$

Moreover for $0 \le \beta < n$, $0 < \alpha p < n - \beta$ *and* $g \in M^{p,\beta}$ *we have*

$$|A_\alpha[g]| \le C(\alpha,\beta,n) M_\beta[g^p]^{\frac{\alpha}{n-\beta}} M[g]^{\frac{n-\beta-\alpha p}{n-\beta}},$$

with

$$C(\alpha,\beta,n,p) \le C(n) \sup\left\{\frac{1}{1-\left(\frac{1}{2}\right)^{\alpha}}, \frac{1}{1-\left(\frac{1}{2}\right)^{\alpha-\frac{n-\beta}{p}}}\right\}.$$

Remark 2.8.5. 1. The second estimate follows from the first: Recall the trivial estimate $g_* \le CM[g]$ and using the Hölder inequality, if $g \in M^{p,\beta}$ then $g \in M^{1,\hat{\beta}}$ with

$$M_{\hat{\beta}}[g] \le C(n) M_\beta[g^p]^{\frac{1}{p}}$$

and $\hat{\beta} - n = \frac{1}{p}(\beta - n)$. Thus for $p \ge 1$, $g \in M^{p,\beta}$ with $\alpha p < n - \beta$ (so that $\alpha < n - \hat{\beta}$) then by the first part of Proposition 2.8.4

$$\begin{aligned} |A_\alpha[g]| &\le C(\alpha,\hat{\beta},n)(M_{\hat{\beta}}[g](x))^{\frac{\alpha}{n-\hat{\beta}}}(g_*(x))^{\frac{n-\hat{\beta}-\alpha}{n-\hat{\beta}}} \\ &\le C(\alpha,\beta,n,p) M_\beta[g^p]^{\frac{\alpha}{n-\beta}} M[g]^{\frac{n-\beta-\alpha p}{n-\beta}}. \end{aligned}$$

Where we know that

$$C(\alpha,\beta,n,p) \le C(n) \sup\left\{\frac{1}{1-\left(\frac{1}{2}\right)^{\alpha}}, \frac{1}{1-\left(\frac{1}{2}\right)^{\alpha-\frac{n-\beta}{p}}}\right\}.$$

2. An important corollary of the first estimate is that for ($\alpha = 1$, $\beta = n-2$) $g \in M^{1,n-2} \cap h^1(\mathbb{R}^n)$ then $g \in H^{-1}(E)$ for any compact $E \subset \mathbb{R}^n$: Let $\tilde{g} = \psi(g-\lambda) + \psi\lambda$ where $\psi \in C_c^\infty$ and $\psi \equiv 1$ on E (so that $\tilde{g} = g$ in E). We have $\psi(g-\lambda) \in \mathscr{H}^1$ (see Section 2.6), and $\lambda\psi \in L^\infty(E)$, so we know $A_1[\tilde{g}] \in L^2(E)$ with ($\tilde{g}$ has the same decay as g)

$$\|A_1[\tilde{g}]\|_{L^2(E)} \le C(\|M_{n-2}[g]\|_{L^\infty(\mathbb{R}^n)}\|g\|_{h^1(\mathbb{R}^n)})^{\frac{1}{2}} = C\|g\|^{\frac{1}{2}}_{M^{1,n-2}}\|g\|^{\frac{1}{2}}_{h^1}.$$

Now set $w = N[\tilde{g}] = \Gamma * \tilde{g}$ where N is the Newtonian potential, we have that $\nabla_i w = \nabla_i N[\tilde{g}] = \nabla_i \Gamma * \tilde{g}$ is an operator of the form A_1 for $a(x) = \frac{x_i}{|x|^n}$ ($\alpha = 1$). Therefore for $\phi \in C_c^\infty(E)$ we can test

$$\int_E g\phi = \int_E (\psi(g-\lambda) + \lambda\psi)\phi = \int_E \Delta w \phi = -\int_E \nabla w . \nabla \phi \le C\|\nabla w\|_{L^2(E)}\|\phi\|_{W^{1,2}(E)}.$$

Thus

$$\|g\|_{H^{-1}(E)} \le C\|g\|_{M^{1,n-2}}^{\frac{1}{2}}\|g\|_{h^1}^{\frac{1}{2}}.$$

This is used to obtain estimate (6.5).

Proof of Proposition 2.8.4. Given Remark 2.8.5 we only need to prove part 1: We split $A_\alpha(g)$ up into its near and far parts using a partition of unity subordinate to dyadic annuli of a chosen modulus δ, more precisely: Let $\theta(x) \in C_c^\infty(B_4 \backslash B_{\frac{1}{2}})$ with $\theta(x) > 0$ for $1 \le |x| \le 2$. Similarly as is done in Semmes [Sem94] we can arrange so that

$$\sum_{j\in\mathbb{Z}} \theta(\delta^{-1}2^{-j}x) = 1$$

for all $x \in \mathbb{R}^n \backslash \{0\}$, moreover we want for our choice of a that $C\int\theta(4x)a(x) = 1$ for some constant C (for reasons that will become apparent below). Notice that $\theta(4\cdot)a(\cdot) \in C_c^\infty(B_1)$ also.

Now define $\eta^j(x) := \delta^{-1}2^{-j}\theta(\delta^{-1}2^{-j}x)a(x)$. Notice that $\delta 2^j\eta^j(x)$ is the piece of a around $\delta 2^{j-1} \le |x| \le \delta 2^{j+2}$, so that

$$A_\alpha(g) = \sum_{j\in\mathbb{Z}} \delta 2^j \eta^j * g = \sum_{j\le 0} \delta 2^j \eta^j * g + \sum_{j\ge 1} \delta 2^j \eta^j * g = I_{inner} + I_{outer}.$$

The intuition here is that we use the decay estimate we have on g in order to deal with I_{outer} and we use the Hardy space qualities of g in order to deal with I_{inner}.

With that in mind we start with estimating I_{inner}. We use the following claim:

$$|\eta^j * g(x)| \le C(\delta^{-1}2^{-j})^{1-\alpha} g_*(x).$$

This is easy enough to see, first of all we remark that in our definition of g_* we choose to use the function $\psi(x) := C\theta(4x)a(x)$, therefore $g_*(x) := \sup_{t>0}|\psi_t * g(x)|$.

$$\begin{aligned}
|\eta^j * g(x)| &= \left|\int_{B_{\delta 2^{j+2}}} \delta^{-1}2^{-j}\theta(\delta^{-1}2^{-j}(x-y))a(x-y)g(y)dy\right| \\
&= \left|\int_{B_{\delta 2^{j+2}}} \delta^{-1}2^{-j}\theta(\delta^{-1}2^{-j}(x-y))a(\delta^{-1}2^{-(j+2)}(x-y))g(y)(\delta^{-1}2^{-(j+2)})^{n-\alpha}dy\right| \\
&= C(\delta^{-1}2^{-j})^{1-\alpha}|\psi_{\delta 2^{j+2}} * g(x)| \le C(\delta^{-1}2^{-j})^{1-\alpha} g_*(x).
\end{aligned}$$

We estimate

$$\begin{aligned}
|I_{inner}| &\leq \sum_{j\leq 0} \delta 2^j |\eta^j * g(x)| \\
&\leq C\sum_{j\leq 0} \delta 2^j (\delta^{-1}2^{-j})^{1-\alpha} g_*(x) \\
&\leq C\frac{1}{1-\left(\frac{1}{2}\right)^{\alpha}}\delta^{\alpha} g_*(x).
\end{aligned}$$

Now we estimate I_{outer}

$$\begin{aligned}
|I_{outer}| &\leq C\sum_{j\geq 1}\int_{\delta 2^{j-1}\leq|x-y|\leq\delta 2^{j+2}} |\theta(\delta^{-1}2^{-j}(x-y))||a(x-y)||g(y)|dy \\
&\leq C\sum_{j\geq 1}\int_{\delta 2^{j-1}\leq|x-y|\leq\delta 2^{j+2}} |x-y|^{\alpha-n}|g(y)|dy \\
&\leq C\sum_{j\geq 1}(\delta 2^{j-1})^{\alpha-n}\int_{|x-y|\leq\delta 2^{j+2}} |g(y)|dy \\
&\leq C\sum_{j\geq 1}(\delta 2^{j-1})^{\alpha-n}(\delta 2^{j+2})^{\beta} M_\beta(g)(x) \\
&\leq C\frac{1}{1-\left(\frac{1}{2}\right)^{\alpha-(n-\beta)}}\delta^{\alpha-(n-\beta)} M_\beta(g)(x).
\end{aligned}$$

Putting together these threads gives us

$$|A_\alpha(g)| \leq C(\alpha,\beta,n)(\delta^\alpha g_*(x) + \delta^{\alpha-(n-\beta)}M_\beta(g)(x)).$$

Setting $\delta = \left(\frac{M_\beta(g)(x)}{g_*(x)}\right)^{\frac{1}{n-\beta}}$ gives

$$|A_\alpha(g)| \leq C(\alpha,\beta,n)(M_\beta(g)(x))^{\frac{\alpha}{n-\beta}}(g_*(x))^{\frac{n-\beta-\alpha}{n-\beta}}.$$

□

Proof of Theorem 2.8.2. Let $f \in M^{p,\beta}$, $p \geq 1$ and $0 < \alpha p < n-\beta$. We will show that

$$\begin{cases}
\|A_\alpha[f]\|^{\tilde{p}}_{L^{\tilde{p}}(B_r(x))} \leq C\|f\|^{\tilde{p}}_{M^{p,\beta}} r^\beta & \text{if } p>1 \\
|\{z\in B_r(x) : |A_\alpha[f](z)|>s\}| \leq C\|f\|^{\tilde{p}}_{M^{1,\beta}} s^{-\tilde{p}} r^\beta & \text{if } p=1
\end{cases}.$$

To that end write $f_r = f\chi_{B_{2r}(x)}$ and $f^r = f - f_r$. For f_r we know that

$$\|f_r\|_{L^p} \leq Cr^{\frac{\beta}{p}}\|f\|_{M^{p,\beta}(E)}$$

thus by Theorem 2.6.1 we have

$$\begin{cases} \|M[f_r]\|_{L^p} \le C(n)\left(\frac{p}{p-1}\right)^{\frac{1}{p}} r^{\frac{\beta}{p}} \|f\|_{M^{p,\beta}(E)} & \text{if } p>1 \\ |\{x : M[f_r](x) > s\}| \le C(n)s^{-1}r^{\beta}\|f\|_{M^{1,\beta}} & \text{if } p=1. \end{cases}$$

Now we can directly apply the second estimate from Proposition 2.8.4 to conclude

$$\begin{cases} \|A_\alpha[f_r]\|^{\tilde{p}}_{L^{\tilde{p}}(B_r(x))} \le C(n,\alpha,\beta,p)^{\tilde{p}} \frac{p}{p-1} \|f\|^{\tilde{p}}_{M^{p,\beta}} r^{\beta} & \text{if } p>1 \\ |\{z \in B_r(x) : |A_\alpha[f_r](z)| > s\}| \le C(n,\alpha,\beta)^{\tilde{p}} \|f\|^{\tilde{p}}_{M^{1,\beta}} s^{-\tilde{p}} r^{\beta} & \text{if } p=1. \end{cases}$$

Where $C(n,\alpha,\beta)$ is the constant appearing in Proposition 2.8.4.

If $z \in B_r(x)$ then

$$\begin{aligned} |A_\alpha[f^r](z)| &\le C\int_{\mathbb{R}^n\setminus B_r(z)} \frac{1}{|z-y|^{n-\alpha}}|f(y)|\,\mathrm{d}y \\ &\le C\sum_{j\ge1}\int_{B_{2^j r}(z)\setminus B_{2^{j-1}r}(z)} \frac{1}{|z-y|^{n-\alpha}}|f(y)|\,\mathrm{d}y \\ &\le C\sum_{j\ge1}(2^j r)^{\alpha-n}\|f\|_{M^{p,\beta}} r^{n-\frac{n-\beta}{p}} \\ &= C\frac{1}{1-\left(\frac{1}{2}\right)^{\alpha-\frac{n-\beta}{p}}}\|f\|_{M^{p,\beta}} r^{\alpha-\frac{n-\beta}{p}} \\ &\le C(n,\alpha,\beta,p)\|f\|_{M^{p,\beta}} r^{\alpha-\frac{n-\beta}{p}}. \end{aligned}$$

Therefore for $p \ge 1$

$$\|A_\alpha[f^r]\|^{\tilde{p}}_{L^{\tilde{p}}(B_r(x))} \le C(n,\alpha,\beta,p)^{\tilde{p}}\|f\|^{\tilde{p}}_{M^{p,\beta}} r^{\beta}.$$

The estimate follows.

□

Chapter 3

Geometric preliminaries

3.1 Vector bundles and connections

Here we remind the reader of some fundamental facts concerning vector bundles over smooth manifolds $\mathscr{M}$ and connections thereon. For a more comprehensive introduction the reader should consult for example [KN63], [Jos11].

Locally, a vector bundle is diffeomorphic to a product of a little piece of $\mathscr{M}$ with a vector space V^k. However the global picture has no such simple description (except in the case of the trivial vector bundle $\mathscr{M} \times V$). Below, and for the remainder of this chapter we will write $V = \mathbb{F}^k$ with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ depending on whether V is real or complex.

Definition 3.1.1. A rank k vector bundle over a smooth manifold $\mathscr{M}$ is a triple $(E, \pi, \mathscr{M})$ where $\pi : E \to \mathscr{M}$ is the projection (a smooth surjective map), E is the total space (another smooth manifold) and $\mathscr{M}$ is the base space. Moreover they must satisfy the following conditions

1. For each $p \in \mathscr{M}$, $\pi^{-1}(p) \subset E$ is a k-dimensional vector space

2. There is a neighbourhood $p \in U_\alpha \subset \mathscr{M}$ and diffeomorphisms ϕ_α such that

$$\phi_\alpha = (\phi_\alpha^1, \phi_\alpha^2) : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{F}^k$$

   and $\phi_\alpha^2 : \{\pi^{-1}(p)\} \to \mathbb{F}^k$ is a linear isomorphism for each $p \in U_\alpha$.

3. The map $\pi : \pi^{-1}(U_\alpha) \to U_\alpha$ is given by the map ϕ_α^1.

Remark 3.1.2. By considering two overlapping domains $p \in U_\alpha \cap U_\beta$ we could consider $\phi_{\alpha\beta} := \phi_\alpha \circ \phi_\beta^{-1} : \{p\} \times \mathbb{F}^k \to \{p\} \times \mathbb{F}^k$ being a linear isomorphism which varies smoothly for $p \in U_\alpha \cap U_\beta$. Therefore given a vector bundle we can define smooth maps known as transition functions $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{F})$ satisfying

$$\phi_{\alpha\alpha}(p) = I_k$$

and

$$\phi_{\alpha\beta}(p)\phi_{\beta\gamma}(p) = \phi_{\alpha\gamma}(p).$$

Given a vector bundle $(E, \pi, \mathscr{M})$ over a manifold $\mathscr{M}$, we say that $v \in \Gamma(E)$ or v is a section of E if it is a smooth map $v : \mathscr{M} \to E$ such that $\pi \circ v(p) = p$ for all $p \in \mathscr{M}$. Thus for each point in our manifold v assigns a value in the vector space $\{\pi^{-1}(p)\}$. For instance a vector field is a section of the tangent bundle $\mathrm{T}\mathscr{M}$. In terms of a local trivialisation over a neighbourhood $U_\alpha \subset \mathscr{M}$ this vector field is given in terms of a smooth map $v_\alpha = \phi_\alpha^2 \circ v : U_\alpha \to \mathbb{F}^k$ and whenever two such domains overlap $U_\alpha \cap U_\beta \neq \emptyset$ then $v_\alpha(p) = \phi_{\alpha\beta}(p)\, v_\beta(p)$.

It is clear that locally we can take a directional derivative of v_α in a direction $X \in \mathrm{T}_p\mathscr{M}$ to give another vector $z_\alpha(p) = \mathrm{d}v_\alpha(X)$, however this directional derivative is not well defined because of the lack of global triviality of our vector bundle: Given a different trivialisation at a point p, let $v_\alpha, v_\beta : U_\alpha \cap U_\beta \to \mathbb{F}^k$ represent v via the trivialisations ϕ_α and ϕ_β respectively. Now we can take derivative $\mathrm{d}v_\alpha(X) = z_\alpha$ and $\mathrm{d}v_\beta(X) = z_\beta$, but in order that this be well defined we must have $z_\alpha = \phi_{\alpha\beta} z_\beta$. This requires that $\phi_{\alpha\beta}$ be locally constant since

$$z_\alpha = \mathrm{d}v_\alpha(X) = \mathrm{d}\phi_{\alpha\beta}(X)\, v_\beta + \phi_{\alpha\beta}\mathrm{d}v_\beta(X) = \mathrm{d}\phi_{\alpha\beta}(X)\, v_\beta + \phi_{\alpha\beta} z_\beta.$$

We say that a vector bundle is flat if the transition charts are locally constant, at which point we can just extend our usual notion of directional (exterior) derivative.

In general, the notion of a connection is what is missing; we are missing the ability to connect nearby fibres of our vector bundle E in a canonical way – and thus have a well defined directional derivative. We in fact have a choice of how we do this. After all of this discussion, lets abstractly define an affine connection on a vector bundle.

Definition 3.1.3. Given a smooth vector bundle $(E, \pi, \mathscr{M})$ a connection is a smooth map

$$\mathrm{d}_E : \Gamma(E) \to \Gamma(\mathrm{T}^*\mathscr{M} \otimes E)$$

and for $v \in \Gamma(E)$ and $X \in \Gamma(\mathrm{T}\mathscr{M})$ we write $D^E_X v = \mathrm{d}_E v(X)$ which denotes the derivative of v in direction X. For all $v, w \in \Gamma(E)$, $X, Y \in \Gamma(\mathrm{T}\mathscr{M})$ and $f \in C^\infty(\mathscr{M})$ we must have

1.
$$D^E_{fX} v = f D^E_X v$$
and
$$D^E_{X+Y} v = D^E_X v + D^E_Y v$$

2.
$$D^E_X f v = \mathrm{d}f(X) v + f D^E_X v$$
and
$$D^E_X (v + w) = D^E_X v + D^E_X w$$
i.e. there is a Leibnitz rule.

Notice that if we were linear over $C^\infty(\mathscr{M})$ in v then we could understand $\mathrm{d}_E \in \Gamma(E \otimes E^* \otimes \mathrm{T}^*\mathscr{M}) = \Gamma(End(E) \otimes \mathrm{T}^*\mathscr{M})$, however this is not the case for a single connection (because we have a Leibnitz rule) but given two connections d^1_E and d^2_E we can check that $\mathrm{d}^1_E - \mathrm{d}^2_E \in \Gamma(End(E) \otimes \mathrm{T}^*\mathscr{M})$. Thus the entire space of connections on a vector bundle can be obtained by picking a fixed connection and translating by sections of $End(E) \otimes \mathrm{T}^*\mathscr{M}$. We remark here that given any vector bundle, there is always a connection (and therefore as many connections as there are sections of $End(E) \otimes \mathrm{T}^*\mathscr{M}$!).

Now, given a local trivialisation of our vector bundle over U_α we know that the exterior derivative defines a connection locally (not globally!), so that any connection on E is locally described by a section of $End(E) \otimes \mathrm{T}^* U_\alpha$ say ω_α. In our fixed trivialisation ϕ_α over U_α, sections are given by maps $v_\alpha : U_\alpha \to \mathbb{F}^k$ and $\omega_\alpha : U_\alpha \to gl(k, \mathbb{F}) \otimes \mathrm{T}^* U_\alpha$ and our connection acts on v by

$$\begin{aligned} \mathrm{d}_E v &= \frac{\partial v^i_\alpha}{\partial x^a} \mathrm{d}x^a \otimes e_i + v^j_\alpha (\omega_\alpha)^i_{ja} \mathrm{d}x^a \otimes e_i \\ &= \mathrm{d}v_\alpha + \omega_\alpha(v_\alpha). \end{aligned} \tag{3.1}$$

On two overlapping neighbourhoods $U_\alpha \cap U_\beta \neq \emptyset$ we can check using this expression that we must have

$$\phi^{-1}_{\alpha\beta} \mathrm{d}\phi_{\alpha\beta} + \phi^{-1}_{\alpha\beta} \omega_\alpha \phi_{\alpha\beta} = \omega_\beta. \tag{3.2}$$

Notice that given any trivialisation $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{F}^k$ we could take any smooth map $P_\alpha : U_\alpha \to GL(k, \mathbb{F})$ and get a new trivialisation by multipying by P in the second variable. In other words for $p \in U_\alpha$ we could consider a new trivialisation $\tilde{\phi}_\alpha(s) = (p, P(p)v)$ with $s \in \pi^{-1}(p)$ and $\phi_\alpha(s) = (p, v)$. We can thus re-write our connection forms ω_α with respect to our new trivialisation (or frame) P_α to give ω_{P_α}. By (3.2) we have that

$$\omega_{P_\alpha} = P_\alpha^{-1} \mathrm{d} P_\alpha + P_\alpha^{-1} \omega_\alpha P_\alpha.$$

Suppose for a moment that our vector bundle is trivial then clearly it is flat. Moreover we could check whether a given connection is just the flat connection: In this case the connection globally takes the form (3.1) and we say that it is flat if there exists a change of frame P such that $\omega_P \equiv 0$. In general we call a connection flat if we can find a local change of frame such that $\omega_{P_\alpha} \equiv 0$ over every trivialisation. It is an excercise to check that the new transition charts $\tilde{\phi}_{\alpha\beta}$ ($\tilde{\phi}_\alpha$ is defined above) are locally constant. Thus a flat vector bundle is one that supports a flat connection, and we can always find trivialisations which makes this connection the exterior derivative.

We can think of our connection as defining a covariant exterior derivative on $E^k := E \otimes \wedge^k \mathrm{T}^* \mathscr{M}$ (so that $E^0 = E$) we define an exterior derivative:

$$\mathrm{d}_{E^k} : \Gamma(E^k) \to \Gamma(E^{k+1})$$

by

$$\mathrm{d}_{E^k}(v \otimes \mu) = \mathrm{d}_E v \wedge \mu + v \otimes \mathrm{d}\mu.$$

Thus, if both the underlying manifold had a metric and our bundle had a fibre metric, then we also have a notion of covariant divergence d_E^* by considering the formal adjoint of the covariant exterior derivative. I.e. we define

$$\mathrm{d}_{E^k}^* : \Gamma(E^{k+1}) \to \Gamma(E^k)$$

by requiring that

$$\int_{\mathscr{M}} \langle \mathrm{d}_{E^k} v, z \rangle = \int_{\mathscr{M}} \langle v, \mathrm{d}_{E^k}^* z \rangle$$

for all compactly supported $v \in \Gamma(E^k)$ and $z \in \Gamma(E^{k+1})$. The inner product here is of course the one induced by the metric on $\mathscr{M}$ and the fibre metric. In the case that $k = 0$ we have

$$\mathrm{d}_E^* z_\alpha = \mathrm{d}^* z_\alpha + \langle \bar{\omega}_\alpha^T, z_\alpha \rangle$$

where $\langle \bar{\omega}_\alpha^T, z_\alpha \rangle$ is an inner product with respect to one forms, coupled with matrix multiplication.

Also in the case that we have a bundle metric, we have the notion of a metric connection, which is required to satisfy (for $v, w \in \Gamma(E)$ and $X \in \Gamma(\mathrm{T}\mathscr{M})$)

$$\mathrm{d}\langle v, w \rangle(X) = \langle D_X^E v, w \rangle + \langle v, D_X^E w \rangle.$$

We leave it to the reader to check that if we take a local trivialisation such that the fibre metric is the standard Euclidean (or Hermitian) metric, then our local connection forms must be skew symmetric (skew-hermitian). In this case we have

$$\mathrm{d}_E^* z_\alpha = \mathrm{d}^* z_\alpha - \langle \omega_\alpha, z_\alpha \rangle.$$

We can also discuss the curvature of a connection $F = \mathrm{d}_{E^1} \circ \mathrm{d}_E : \Gamma(E) \to \Gamma(E^2)$, locally we have

$$F = \mathrm{d}\omega + [\omega \wedge \omega].$$

One can check that F is tensorial thus $F \in \Gamma(E^* \otimes E \otimes \wedge^2 \mathrm{T}^* \mathscr{M}) = \Gamma(End(E) \otimes \wedge^2 \mathrm{T}^* \mathscr{M})$. It is clear from the definition that F is measuring how far d_E is from being the exterior derivative, thus we have the following well known theorem.

Theorem 3.1.4. $F \equiv 0$ *if and only if the connection is flat.*

In all of the above we have not distinguished between a real or complex vector bundle, and the base manifold was always real. Now we consider a complex base manifold (one whose cross-over charts are holomorphic). In this case we can consider everything as above, but we can ask whether a vector bundle is holomorphic rather than just being flat. We say a vector bundle is holomorphic if the transition charts are locally holomorphic and a connection is holomorphic if there is always a change of frame such that $\omega_{P_\alpha}^{(0,1)} \equiv 0$. We can check that a vector bundle is holomorphic if and only if it supports a holomorphic connection. We have the following, due to Koszul-Malgrange [KM58].

Theorem 3.1.5. $F^{(0,2)} \equiv 0$ *if and only if the connection is holomorphic.*

3.2 Non smooth connections

Now we revert to using a purely local set up for a connection on a vector bundle over a ball. Thus our vector bundle is a product $B_1 \times \mathbb{F}^k$ and a connection is defined completely by the connection forms. Moreover we are free to change frame over our trivial bundle, therefore changing our connection forms along with it. We ask therefore whether there is a desirable frame in which to express our connection? If the connection is flat then clearly we would like to use the frame that sets our connection forms to be zero, but in general we cannot expect to have a flat connection! It turns out that there is a frame in which to express a given connection locally (a Coulomb frame) which has nice properties.

We will also work with connections that are not smooth. In this case the connection is defined simply by a map $\omega : B_1 \to gl(k,\mathbb{F}) \otimes \wedge^1 \mathrm{T}^* B_1$ and we say that our connection is in L^2 if this map is square integrable. Of course we will use the standard metric to compute the size of ω at any point, however for a general vector bundle we would first impose a fibre metric, then we could always find (locally) a trivialisation that made the metric the standard Euclidean ($\mathbb{F} = \mathbb{R}$) or Hermitian ($\mathbb{F} = \mathbb{C}$) inner product. We could define the L^2 norm over the whole manifold, which would depend on a trivialisation and the metric. If the manifold were closed then any two such choices would yield equivalent norms.

We also mention here that defining spaces of non-smooth sections of vector bundles is trivial since locally these sections are described by maps taking values in a vector space (and therefore there is a global coordinate system) – for maps $f : \mathcal{M} \to \mathcal{N}$ between two Riemannian manifolds this is much more difficult. Thus whichever norm we wish to take for our section can be computed piece-meal over our base manifold and stitched together. As stated earlier a connection is not defined by a section of a vector bundle, however we know that the difference of two connections is a section of $\mathrm{T}^*\mathcal{M} \otimes End(E)$. We can therefore define a non-smooth connection on a vector bundle over a manifold by fixing a smooth reference connection D, and letting the space of non-smooth connections X (for example, those that are square-integrable) be given by

$$X := \{D + S : S \in \Gamma(\mathrm{T}^*\mathcal{M} \otimes End(E)) \text{ is square integrable}\}.$$

Since we are just working locally we can avoid these subtleties and use the standard metric in order to compute the 'size' of our connection (the smooth reference connection will be the exterior derivative). We will be dealing with metric connections, so locally our connection forms look like $\omega \in L^2(B_1, \mathfrak{g}_{\mathbb{F}} \otimes \wedge^1 \mathrm{T}^* B_1)$ where $\mathfrak{g}_{\mathbb{F}}$ denotes the Lie algebra of $\mathfrak{G}_{\mathbb{F}} = SO(k,\mathbb{R})$ or $U(k)$ respectively as $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Thus ω is a skew-symmetric (skew-hermitian) matrix of one forms when $\mathbb{F} = \mathbb{R}$ ($\mathbb{C}$).

A metric change of frame now is given by a map (let us say smooth for now) $P : B_1 \to \mathfrak{G}_{\mathbb{F}}$. Thus we could consider all possible changes of frame and local 'versions' of ω along with it. In fact the reason we consider metric connections is so that the changes of frame preserve the square integrability of our connection (since P carries a natural L^∞ bound in this case) i.e.

$$P^{-1}\mathrm{d}P + P^{-1}\omega P \in L^2$$

when $\omega \in L^2$. It turns out that a useful frame (first considered by Karen Uhlenbeck [Uhl82]) to use is one which is a critical point of the following functional (a critical point P is known as a Coulomb gauge or frame)

$$L_\omega(P) := \int_{B_1} |P^{-1}\mathrm{d}P + P^{-1}\omega P|^2\,\mathrm{d}x$$

in general of course we could only expect $P \in W^{1,2}(B_1, \mathfrak{G}_{\mathbb{F}})$, which makes the L^∞ bound even more important. A proof of the following in the case that $\mathbb{F} = \mathbb{R}$ can be found in [Sch10], however the proof extends trivially to the case that $\mathbb{F} = \mathbb{C}$.

Theorem 3.2.1. *There exists a minimiser $P \in W^{1,2}(B_1, \mathfrak{G}_{\mathbb{F}})$ of L_ω, moreover any critical point of L_ω satisfies*

$$\mathrm{d}^*(P^{-1}\mathrm{d}P + P^{-1}\omega P) = 0.$$

In the case that P is a minimiser we trivially have

$$\|P^{-1}\mathrm{d}P + P^{-1}\omega P\|_{L^2(B_1)} \leq \|\omega\|_{L^2(B_1)}$$

and

$$\|\mathrm{d}P\|_{L^2(B_1)} \leq 2\|\omega\|_{L^2(B_1)}.$$

Remark 3.2.2. An immediate corollary of this theorem is that there exists some $\xi \in W^{1,2}(B_1, \mathfrak{g}_{\mathbb{F}} \otimes \wedge^2 \mathrm{T}^*\mathbb{R}^n)$ (defined up to a co-closed two-form) such that

$$P^{-1}\mathrm{d}P + P^{-1}\omega P = \mathrm{d}^*\xi.$$

When P is a minimiser we thus have

$$\|P\|_{W^{1,2}(B_1)} + \|\mathrm{d}^*\xi\|_{L^2(B_1)} \leq C\|\omega\|_{L^2(B_1)}.$$

We also remark here that since ω is really just a one-form in L^2 we can do a Hodge decomposition (see appendix A), comparing this we could consider a change of frame as being a non-linear Hodge decomposition. Indeed the reader can check that when $k = 2$ and $\mathbb{F} = \mathbb{R}$ this is indeed a standard Hodge decomposition (similarly when $k = 1$ and $\mathbb{F} = \mathbb{C}$ since $U(1) = SO(2, \mathbb{R})$).

In general and in higher dimensions we will need further conditions on ξ in order to work with them. We have the following Lemma due to Rivière-Struwe-Meyer ([RS08], [MR03]) the proof of which resembles more closely Uhlenbeck's [Uhl82] original idea.

Theorem 3.2.3. *Let $\omega \in M^{2,n-2}(B_1, \mathfrak{g}_{\mathbb{R}} \otimes \wedge^1 \mathrm{T}^*\mathbb{R}^n)$, then there exists an $\varepsilon = \varepsilon(n,k)$ such that whenever*

$$\|\omega\|_{M^{2,n-2}(B_1)} \leq \varepsilon$$

then we can find maps $P \in W^{1,2}(B_1, \mathfrak{G}_{\mathbb{R}})$ *and* $\eta \in W_0^{1,2}(B_1, \mathfrak{g}_{\mathbb{R}} \otimes \wedge^2 \mathrm{T}^* \mathbb{R}^n)$ *such that*

$$P^{-1}\mathrm{d}P + P^{-1}\omega P = \mathrm{d}^*\eta$$

and

$$\mathrm{d}\eta = 0.$$

Moreover there exists $C = C(n,k) < \infty$ *such that*

$$\|\nabla P\|_{M^{2,n-2}(B_1)} + \|\nabla\eta\|_{M^{2,n-2}(B_1)} \le C\|\omega\|_{M^{2,n-2}(B_1)}.$$

Remark 3.2.4. We remark that the proof still works in the complex case, so we could have stated this Theorem for $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{G}_{\mathbb{C}}$.

Moreover this theorem allows for a weak version of Theorem 3.1.4; if $F = 0$ in a weak sense and $\|\omega\|_{M^{2,n-2}}$ is small enough then $\eta \equiv 0$.

Of course when $n = 2$ the Morrey space $M^{2,n-2} = L^2$. An important discovery of Rivière [Riv07] is that one can perturb this change of frame P in two dimensions in order to obtain a more useful change of frame, however one must leave the compact group $\mathfrak{G}_{\mathbb{F}}$.

Theorem 3.2.5. *Here* $B_1 \subset \mathbb{R}^2$*. Let* $\omega \in L^2(B_1, \mathfrak{g}_{\mathbb{R}} \otimes \wedge^1 \mathrm{T}^* \mathbb{R}^2)$*, then there exists an* $\varepsilon = \varepsilon(k)$ *such that whenever*

$$\|\omega\|_{L^2(B_1)} \le \varepsilon$$

then we can find maps $A \in L^\infty \cap W^{1,2}(B_1, GL(k,\mathbb{R}))$ *and* $B \in W_0^{1,2}(B_1, gl(k,\mathbb{R}) \otimes \wedge^2 \mathrm{T}^* \mathbb{R}^2)$ *such that*

$$A^{-1}\mathrm{d}A + A^{-1}\omega A = \mathrm{d}^* BA.$$

Moreover there exists a $C = C(k) < \infty$ *such that*

$$\|\mathrm{dist}(\mathrm{A}, \mathfrak{G}_{\mathbb{R}})\|_{\mathrm{L}^\infty(\mathrm{B}_1)} + \|\nabla \mathrm{A}\|_{\mathrm{L}^2(\mathrm{B}_1)} + \|\nabla \mathrm{B}\|_{\mathrm{L}^2(\mathrm{B}_1)} \le \mathrm{C}\|\omega\|_{\mathrm{L}^2(\mathrm{B}_1)}.$$

Remark 3.2.6. Again we could replace $\mathbb{R}$ with $\mathbb{C}$ and the result remains true. We could use different notation since $B_1 \subset \mathbb{R}^2$, we can confuse B with $*B$ and we would have

$$A^{-1}\nabla A + A^{-1}\omega A = \nabla^\perp BA.$$

In higher dimensions a similar result is possible, due to Laura Keller [Kel10], where one assumes that ω is small in a slightly more restrictive Besov-Morrey space related to $M^{2,n-2}$.

Example 3.2.7. Some important examples for us to consider will be covariant divergence-free sections which are square integrable $v \in L^2(B_1, \mathbb{F}^k \otimes \wedge^1 \mathrm{T}^*\mathbb{R}^n)$ with an L^2 connection. We will assume that locally we have an orthogonal or hermitian trivialisation giving $\omega \in L^2(B_1, \mathfrak{g}_{\mathbb{F}} \otimes \mathrm{T}^*\mathbb{R}^n)$. Thus v solves

$$\mathrm{d}_E^* v = \mathrm{d}^* v - \langle \omega, v \rangle = 0$$

where ω are our local connection forms.

1. Suppose we could find $P : U \to \mathfrak{G}_{\mathbb{F}}$ such that

$$P^{-1}\mathrm{d}P + P^{-1}\omega P = 0,$$

thus

$$\begin{aligned} \mathrm{d}^*(P^{-1}v) &= P^{-1}\langle \omega, v \rangle - \langle \mathrm{d}P^{-1}, v \rangle \\ &= \langle P^{-1}\omega P, P^{-1}v \rangle + \langle P^{-1}\mathrm{d}P, P^{-1}v \rangle \\ &= 0. \end{aligned}$$

Therefore there is a trivialisation such that v is genuinely divergence-free moreover there is some $\mu \in W^{1,2}(B_1, \mathbb{F}^k \otimes \wedge^2 \mathrm{T}^*\mathbb{R}^n)$ such that

$$P^{-1}v = \mathrm{d}^*\mu.$$

Now suppose that v is locally exact, so that $v = \mathrm{d}u$ for some $u \in W^{1,2}(B_1, \mathbb{F}^k)$ then we would have

$$-\Delta u = \mathrm{d}^*\mathrm{d}u = *(\mathrm{d}P \wedge \mathrm{d} * \mu) \in \mathscr{H}^1.$$

2. Let P be a Coulomb gauge (using Theorem 3.2.1) then

$$P^{-1}\mathrm{d}P + P^{-1}\omega P = \mathrm{d}^*\xi.$$

Thus we have

$$\mathrm{d}_E^*\mathrm{d}u = \mathrm{d}^*\mathrm{d}u - \langle \omega, \mathrm{d}u \rangle = 0$$

and

$$\begin{aligned} \mathrm{d}^*(P^{-1}\mathrm{d}u) &= \langle \mathrm{d}^*\xi, P^{-1}\mathrm{d}u \rangle \\ &= *(\mathrm{d} * \xi \wedge P^{-1}\mathrm{d}u) \end{aligned}$$

which is nearly in $\mathscr{H}^1$ (up to multiplication by P^{-1}).

3. When $n = 2$, using Rivière's perturbation of the Coulomb gauge we would actually have

$$\mathrm{d}^*(A^{-1}\mathrm{d}u) = *(\mathrm{d} * B \wedge \mathrm{d}u) \in \mathscr{H}^1.$$

It turns out that the quantities arising above which are nearly in $\mathscr{H}^1$ are enough to conclude higher regularity of solutions $v = \mathrm{d}u$.

3.3 Complex domains

Here we restrict our domain manifold to a Riemannian surface $\mathscr{M}^2$ and consider a connection on a real or complex vector bundle E over our surface. Since $\mathscr{M}$ is two dimensional we can view it as a Riemann surface with the conformal structure compatible with the underlying metric. Setting $E^{(p,q)} := E \otimes \wedge^{(p,q)}\mathrm{T}^*_{\mathbb{C}}\mathscr{M}$, as above we can consider our connection to induce a covariant exterior derivative $\mathrm{d}_E : E^{(p,q)} \to E^{(p+1,q+1)}$. Moreover we could consider related ∂_E and $\overline{\partial}_E$. In particular we have a $\overline{\partial}$-operator

$$\overline{\partial}_{E^{(1,0)}} : \Gamma(E^{(1,0)}) \to \Gamma(E^{(1,1)})$$

locally given by

$$\overline{\partial}_E(\alpha) = \overline{\partial}\alpha + \omega^{\overline{z}} \wedge \alpha$$

where $\omega^{\overline{z}} := \frac{1}{2}(\omega^x + i\omega^y)\mathrm{d}\overline{z}$ and $\omega = \omega^x \mathrm{d}x + \omega^y \mathrm{d}y$ are the connection forms. I.e. ω^x, ω^y are maps into $\mathfrak{g}_{\mathbb{F}}$ and $z = x + iy$ is our local coordinate.

We can ask if we can find a frame that allows us to put a holomorphic structure on our vector bundle. If our vector bundle is real we shall first need to complexify each fibre and seek a frame for the complexified bundle – unless that connection is flat! We have the following theorem which is a weak version of Theorem 3.1.5 (since $\dim_{\mathbb{C}}(\mathscr{M}) = 1$ we always have $F^{(0,2)} \equiv 0$), a proof of which can be found in [Hél02].

Theorem 3.3.1. *Let $\omega^{\overline{z}} \in L^{2,1}(D, gl(k,\mathbb{C}) \otimes \wedge^{(1,0)}\mathrm{T}^*_{\mathbb{C}}\mathbb{C})$ then there exists $\varepsilon > 0$ such that whenever $\|\omega\|_{L^{2,1}(D)} \leq \varepsilon$ there is a $P \in W^{1,(2,1)}(D, GL(k,\mathbb{C}))$ such that*

$$\overline{\partial}P = -\omega^{\overline{z}}P$$

and

$$\|\mathrm{dist}(\mathrm{P},\mathrm{Id})\|_{\mathrm{L}^\infty(\mathrm{D})} \leq \frac{1}{3}.$$

Remark 3.3.2. P forces a holomorphic structure on E. Moreover $P \in C^0$ since $\overline{\partial} P \in L^{2,1}$. Also notice that we could write P as a solution to

$$(P^{-1}\mathrm{d}P + P^{-1}\omega P)^{(0,1)} = 0.$$

Example 3.3.3. Suppose that $\alpha \in \Gamma(E^{(1,0)})$ solves

$$\overline{\partial}_{E^{(1,0)}}\alpha = 0$$

and we can locally find a change of frame P as in Theorem 3.3.1. Then we have

$$\overline{\partial}(P^{-1}\alpha) = 0$$

locally. Thus we can think of α as being holomorphic via the frame P.

3.4 Harmonic map regularity theory

Here we give a very brief overview of some well-known results for harmonic map regularity theory and some recent generalisations and improvements. Given Riemannian manifolds $(\mathscr{M}, g)$, $(\mathscr{N}, h)$ and maps between them $u : \mathscr{M} \to \mathscr{N}^N$ we consider the Dirichlet energy

$$E(u) = \int_{\mathscr{M}} |\mathrm{d}u|^2 \mathrm{d}V_m,$$

$\mathrm{d}u \in \Gamma(\mathrm{T}^*\mathscr{M} \otimes u^*\mathrm{T}\mathscr{N})$ and the norm is with respect to the metrics induced by g and h. Harmonic maps are the critical points with respect to this energy, with which, one can compute the Euler-Lagrange equations to give (take a one-parameter family of variations $u_t(x) = Exp_{u(x)}(tv(x))$ for a compactly supported $v \in \Gamma(u^*\mathrm{T}\mathscr{N})$)

$$\tau(u) = \mathrm{d}^*_{u^*\mathrm{T}\mathscr{N}}\mathrm{d}u = 0$$

where $\mathrm{d}^*_{u^*\mathrm{T}\mathscr{N}}$ is the induced covariant divergence with respect to the pull back of the Levi-Civita connection on $u^*\mathrm{T}\mathscr{N}$. We call $\tau(u) \in \Gamma(u^*\mathrm{T}\mathscr{N})$ the tension field. Thus in essence we are dealing with a divergence type problem '$\mathrm{d}u$ is divergence free'-compare with the case that $\mathscr{N} = \mathbb{R}^N$. Locally u solves

$$\Delta_g u^i + g^{ab}\Gamma^i_{jk}(u)\frac{\partial u^j}{\partial x^a}\frac{\partial u^k}{\partial x^b} = 0$$

where $\Delta_g = \mathrm{d}^*\mathrm{d}$ is the Laplace-Beltrami operator. We call $u : \mathcal{M} \to \mathcal{N}$ a harmonic map if it solves the Euler-Lagrange equations. Here of course Γ denotes the Christoffel symbols of the Levi-Civita connection on $\mathcal{N}$.

If $\mathcal{M}$ is two dimensional then we can pick a local isothermal coordinate $z = x + iy$ and we can check that a harmonic map u also solves

$$\overline{\partial}_{u^*\mathrm{T}\mathcal{N}}\partial u = 0$$

where we have complexified the domain, $\partial u = \frac{\partial u}{\partial z}\mathrm{d}z$ and $\overline{\partial}_{u^*\mathrm{T}\mathcal{N}}$ is the induced $\overline{\partial}$-operator associated with the pull back Levi-Civita connection on $u^*\mathrm{T}\mathcal{N}$. In coordinates we have

$$\overline{\partial}\partial u^i + \Gamma^i_{jk}\overline{\partial}u^k \wedge \partial u^j = 0$$

(remember $\Gamma^i_{jk} = \Gamma^i_{kj}$). Thus in two dimensions we have that ∂u is holomorphic. Another way of seeing that we should expect that ∂u is holomorphic is by noticing that $\mathrm{d}u$ solves both

$$\mathrm{d}^*_{u^*\mathrm{T}\mathcal{N}}\mathrm{d}u = 0$$

and

$$\mathrm{d}_{u^*\mathrm{T}\mathcal{N}^1}\mathrm{d}u = 0.$$

Compare this with the flat case $\mathcal{N} = \mathbb{R}^N$.

We have been intentionally vague about our solutions and their regularity: In order to write down such PDE we have assumed that u is at least continuous (so that we can remain in a coordinate neighbourhood of $\mathcal{N}$). However the Dirichlet energy requires only that u have a derivative of some kind and that this derivative be square integrable.

Thus we wish to consider maps $u \in W^{1,2}(\mathcal{M},\mathcal{N})$ as admissible for $E(u)$. Unfortunately the definition of this space is rather unhelpful (one must be careful unless $\dim(\mathcal{M}) = 1$). In order to overcome this difficulty we assume that $\mathcal{N}^N$ (which we can even assume is only C^1-regular) is isometrically embedded into some Euclidean space $\mathbb{R}^m$ and we define

$$W^{1,2}(\mathcal{M},\mathcal{N}) := \{u \in W^{1,2}(\mathcal{M},\mathbb{R}^m) : u(x) \in \mathcal{N} \text{ almost everywhere}\}.$$

In fact, in the case that $\dim(\mathcal{M}) = 2$ we know that smooth functions are dense in this space, however this is not the case in higher dimensions. It is also known that this definition is independent of the isometric embedding of $\mathcal{N}$-see [Hél02, Lemma 1.4.3]. We are now in a position to discuss weakly harmonic maps $u \in W^{1,2}(\mathcal{M},\mathcal{N})$ for closed C^2 manifolds $\mathcal{N}$ isometrically embedded in $\mathbb{R}^m$. Following [Hél02, Definition 1.4.9] we know that there is some projection P from a tubular neighbourhood of $\mathscr{N}$, $V_\delta\mathscr{N}$, back into $\mathscr{N}$ and we say that $u \in W^{1,2}(\mathscr{M},\mathscr{N})$ is weakly harmonic if

$$\lim_{t\to 0}\frac{E(P(u+tv))-E(u)}{t}=0$$

for all $v \in W_0^{1,2}\cap L^\infty(\mathscr{M},\mathbb{R}^m)$. We require that $v\in L^\infty$ so that we can ensure $u+tv\in V_\delta\mathscr{N}$ for small enough t. One can also check that this definition is independent of the isometric embedding of $\mathscr{N}$.

From [Hél02, Lemma 1.4.10] we know that this is equivalent to u being a distributional solution to

$$\Delta_g u + g^{ab}\mathscr{A}(u)\left(\frac{\partial u}{\partial x^a},\frac{\partial u}{\partial x^b}\right)=0$$

and by approximation we have that for all $v\in W_0^{1,2}\cap L^\infty(\mathscr{M},\mathbb{R}^m)$

$$\int_{\mathscr{M}}\left[(\mathrm{d}u,\mathrm{d}v)_g + g^{ab}\mathscr{A}(u)\left(\frac{\partial u}{\partial x^a},\frac{\partial u}{\partial x^b}\right)v\right]\mathrm{d}V_{\mathscr{M}}=0,$$

$\mathscr{A}$ is the second fundamental form of $\mathscr{N}$, i.e. $\mathscr{A}\in\Gamma(\mathrm{T}^*\mathscr{N}\otimes\mathrm{T}^*\mathscr{N}\otimes\mathrm{N}\mathscr{N})$ and for $X,Y\in\Gamma(\mathrm{T}\mathscr{N})$, $\mathscr{A}(X,Y)=(\mathrm{d}X(Y))^\perp$ where we first extend X and Y arbitrarily to $V_\delta\mathscr{N}$ (in order to differentiate) and $()^\perp$ is the projection onto the normal bundle – one can show that this is independent of the extension of X, Y and that $\mathscr{A}$ is symmetric. We can also extend $\mathscr{A}$ and consider it to be a map $\mathscr{A}: V_\delta\mathscr{N}\to\mathrm{T}^*\mathbb{R}^m\otimes\mathrm{T}^*\mathbb{R}^m\otimes\mathrm{T}\mathbb{R}^m$ by first projecting onto $\mathrm{T}\mathscr{N}$ – V_δ is a tubular neighbourhood of $\mathscr{N}$. When $\mathscr{N}$ is C^2, the components of $\mathscr{A}$ will be continuous and therefore bounded ($\mathscr{N}$ is closed). Thus we write $\mathscr{A}(u)\left(\frac{\partial u}{\partial x^a},\frac{\partial u}{\partial x^b}\right)^i=\mathscr{A}^i_{jk}(u)\frac{\partial u^j}{\partial x^a}\frac{\partial u^k}{\partial x^b}$. Another way of writing $\mathscr{A}$ locally is to use a local orthonormal frame $\{\nu_i\}_{i=m-N}^m$ for $\mathrm{N}\mathscr{N}$ (extended arbitrarily to $V_\delta\mathscr{N}$), and we would have

$$\mathscr{A}(z)(X,Y)=\sum_k(X,\mathrm{d}\nu_k(Y))\nu_k(z)=\sum_{k,s}X^s(z)Y^l(z)\frac{\partial\nu_k^s(z)}{\partial z^l}\nu_k(z)$$

so that

$$\mathscr{A}(u)\left(\frac{\partial u}{\partial x^a},\frac{\partial u}{\partial x^b}\right)^i=\sum_{k,s}\frac{\partial\nu_k^s(u)}{\partial x^a}\frac{\partial u^s}{\partial x^b}\nu_k^i(u).$$

Thus if $\mathrm{N}\mathscr{N}$ were trivial we could globally define $\mathscr{A}$ in such a way.

We are now in a position to discuss the regularity theory of Harmonic maps. A naive glance at the Euler-Lagrange equations suggests that classical regularity theory does not help – *a-priori* a harmonic map $u\in W^{1,2}(\mathscr{M},\mathscr{N})$ is a weak solution to

$$\Delta_g u=-g^{ab}\mathscr{A}(u)\left(\frac{\partial u}{\partial x^a},\frac{\partial u}{\partial x^b}\right)\in L^1$$

thus we can only conclude that $\nabla u \in L^{\frac{n}{n-1},\infty}$ for which L^2 is a strict subset. However it is known that any continuous weakly harmonic map is as smooth as the data allows (see [Jos11]), so the goal in this case is to prove continuity, from which the full regularity can be obtained.

3.4.1 Two dimensions

In two dimensions $W^{1,2}$ maps are very nearly bounded (and continuous). In fact if $\nabla u \in L^{2,1}$ then it is continuous (Lemma 2.7.3). In two dimensions the full regularity for weakly harmonic maps was proved by Frederic Helein [Hél91a] using the beautiful method of moving frames. The idea was to express our map with respect to a moving tangent frame on $\mathcal{N}$, and pick this frame suitably enough to enable a better regularity theory. Unsurprisingly there are one or two issues with this notion, not least the existence of a global tangent frame on $\mathcal{N}$ (we cannot work locally on $\mathcal{N}$ since we do not know that u is continuous). However if $\mathcal{N}$ is sufficiently regular (C^4) it is possible to isometrically embed $\mathcal{N}$ into a torus $\tilde{\mathcal{N}}$, so that any harmonic map $u : \mathcal{M} \to \mathcal{N}$ lifts to a harmonic map $u : \mathcal{M} \to \tilde{\mathcal{N}}$ [Hél02]. This trick trivialises the tangent bundle $\mathrm{T}\tilde{\mathcal{N}}$, thus one can pick a global frame P in which to express any section of $\mathrm{T}\tilde{\mathcal{N}}$. In the case of harmonic maps the bundle we are interested in is the pull back bundle $u^*\mathrm{T}\tilde{\mathcal{N}}$, and we can express our tension field $\tau(u)$ with the frame $P(u)$. The improvements one obtains in using a Coulomb gauge are of tiny proportions, however they allow one to go from being nearly bounded, to continuous which is a large step! Effectively it comes down to objects lying in $\mathcal{H}^1$ instead of L^1 meaning that derivatives of certain objects lie in $L^{2,1}$ instead of just L^2. For further details see [Hél02, Chapter 4].

This has more recently been generalised by Tristan Riviere [Riv07] to include a much wider class of variational problems and more general PDE (and even harmonic maps when the target is only C^2). In particular given any conformally invariant elliptic Lagrangian, that is quadratic and coercive in the gradient, Rivière observed that the Euler Lagrange equations take the form of a covariant divergence problem with metric connection forms, and can be written in divergence form. This required a perturbation of the Coulomb gauge – Theorem 3.2.3. One of the main results from [Riv07] is the following

Theorem 3.4.1. *Given $u \in W^{1,2}(B_1, \mathbb{R}^m)$ weakly solving*

$$-\Delta u = \omega . \nabla u \tag{3.3}$$

for $\omega \in L^2(B_1, so(m) \otimes \wedge^1 \mathrm{T}^* \mathbb{R}^2)$ then $u \in C^0$.

It is clear that this PDE is critical ($\Delta u \in L^1$), however using a perturbation of the Coulomb gauge Theorem 3.2.5 we see that (when $\|\omega\|_{L^2}$ is small enough)

$$\mathrm{d}^*(A\mathrm{d}u) = \mathrm{d}^* B.\mathrm{d}u = *(\mathrm{d} * B \wedge \mathrm{d}u) \in \mathscr{H}^1$$

and

$$\mathrm{d}(A\mathrm{d}u) = \mathrm{d}A \wedge \mathrm{d}u \in \mathscr{H}^1$$

from which we can conclude $u \in W^{2,1}_{loc} \hookrightarrow C^0$. In particular he also observed that weakly harmonic maps in two dimensions solve (3.3), this follows by first choosing isothermal coordinates and setting

$$\omega^i_j = (\mathscr{A}^i_{jk}(u) - \mathscr{A}^j_{ik}(u))\mathrm{d}u^k.$$

Thus a weakly harmonic map solves

$$-\Delta u^i = \omega^i_j.\mathrm{d}u^j = \mathscr{A}^i_{jk}(u)\mathrm{d}u^k.\mathrm{d}u^j$$

since $\sum_j \mathscr{A}^j_{ik}(u)\mathrm{d}u^j = 0$. Moreover it is also shown that critical points to a much larger class of conformally invariant variational problems satisfy (3.3) and that solutions can be written in divergence form.

We remark here that if the normal bundle of $\mathscr{N}$ is trivial then we could let

$$\omega^i_j = \sum_k \mathrm{d}\nu^i_k(u)\nu^j_k(u) - \mathrm{d}\nu^j_k(u)\nu^i_k(u).$$

3.4.2 Higher dimensions

In higher dimensions it is known that there are weakly harmonic maps that are nowhere smooth (due to Tristan Rivière [Riv92]) i.e. that there is no open neighbourhood for which the function is continuous. In this case there can be no regularity theory for weakly harmonic maps, however there are an important class of harmonic maps, 'weakly stationary' harmonic maps, for which there is some regularity results. Weakly stationary harmonic maps are weakly harmonic maps that are also critical points with respect to variations in the domain. If $u \in W^{1,2}$ is a weakly stationary harmonic map then it is weakly harmonic and it satisfies a monotonicity formula:

$$\rho_x(r) = r^{2-n}\int_{B_r(x)} |\nabla u|^2$$

is a monotone increasing function of r. Thus in particular we have $\nabla u \in M^{2,n-2}$ and using a moving frame technique it is possible to show that whenever $\|\nabla u\|_{M^{2,n-2}}$ is small enough and $\mathcal{N}$ is sufficiently regular, then u is Hölder continuous – initially due to Bethuel [Bet93], Evans [Eva91]. Using this result one can show that a weakly stationary map is smooth away from a closed singular set S with $\mathcal{H}^{n-2}(S) = 0$ where $\mathcal{H}^{n-2}$ is the $n-2$ dimensional Hausdorff measure. In the case of minimising maps it is known that $\dim_{\mathcal{H}}(S) \leq n-3$ [SU82].

Similarly as in the two dimensional case, Rivière-Struwe generalised the weakly stationary harmonic map equation to include a wider variety of critical PDE, and using Coulomb gauge methods they showed that one can prove Hölder regularity.

Chapter 4

Rivière's equation in two dimensions

4.1 Introduction

Suppose $u \in W^{1,2}(B_1, \mathbb{R}^m)$ is a weak solution to

$$-\Delta u = \omega.\nabla u \tag{4.1}$$

where here and throughout chapter B_1 is the unit disc in $\mathbb{R}^2$, $\omega \in L^2(B_1, so(m) \otimes \wedge^1 \mathrm{T}^*\mathbb{R}^2)$, and we are using the notation $(\omega.\nabla u)^i = \langle \omega^i_j, \nabla u^j \rangle$. This equation, first considered in this generality by Rivière [Riv07], generalises a number of interesting equations appearing naturally in geometry, including the harmonic map equation, the H-surface equation and, more generally, the Euler-Lagrange equation of any conformally invariant elliptic Lagrangian which is quadratic in the gradient. A central issue is the regularity of u implied by virtue of it satisfying the equation (4.1). *A priori*, the right-hand side of the equation looks like quite a general L^1 function, and standard elliptic regularity theory does not seem to help. However, Rivière [Riv07] showed that any solution must necessarily be continuous and even in $W^{2,1+\epsilon}$ for some $\epsilon > 0$ [Riv09], thus generalising the famous regularity theory of Hélein [Hél91a], for example. In most known interesting special cases of this equation, one happens to know that $|\omega|$ can be estimated linearly in terms of $|\nabla u|$, i.e. we have $|\omega.\nabla u| \leq C|\nabla u|^2$ and then a standard bootstrapping argument can be applied to improve the regularity of u to $W^{2,p}$ for any $p < \infty$. Moreover typically in such cases ω can be viewed as a smooth function of u and ∇u, at which point we can conclude that u is smooth using Schauder theory (and the embedding $W^{2,p}(\mathbb{R}^2) \hookrightarrow C^{1,1-\frac{2}{p}}(\mathbb{R}^2)$ for $p > 2$).

In this chapter we investigate what sort of regularity properties we can deduce for solutions of the general equation (4.1), and even more general inhomogeneous equations with the same special structure. It is easy to convince oneself that it is unreasonable to expect regularity better than $W^{2,2}$ in general. However, we will show that we *do* have regularity up to this level, or the best possible regularity when there is an inhomogeneity.

4.2 Results

Theorem 4.2.1. *Suppose that $u \in W^{1,2}(B_1, \mathbb{R}^m)$ is a weak solution on the unit disc in $\mathbb{R}^2$ to*

$$-\Delta u = \omega.\nabla u + f, \qquad f \in L^p(B_1, \mathbb{R}^m) \tag{4.2}$$

where $\omega \in L^2(B_1, so(m) \otimes \wedge^1 \mathrm{T}^* \mathbb{R}^2)$ and $p \in (1,2)$. Then $u \in W^{2,p}_{loc}(B_1)$. In particular, if $f \equiv 0$, then $u \in W^{2,p}_{loc}$ for all $p \in [1,2)$ and $u \in W^{1,q}_{loc}$ for all $q \in [1,\infty)$.

Moreover, for $U \subset\subset B_1$, there exist $\eta_0 = \eta_0(p,m) > 0$ and $C = C(p,m,U) < \infty$ so that if $\|\omega\|_{L^2(B_1)} \leq \eta_0$ then

$$\|u\|_{W^{2,p}(U)} \leq C(\|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)}). \tag{4.3}$$

This theorem omits the borderline case $p = 2$ for good reason; even in the case that $f \equiv 0$, one can find solutions so that u is neither $W^{2,2}$ nor Lipschitz. Moreover, examples with $f \equiv 0$ show that the first derivatives of u need not even lie in BMO, and (consequently) the second derivatives need not even lie in the Lorentz space $L^{2,\infty}$ – see section 4.3.

As a corollary of our theorem, we see that $f \in L^p$ implies that u lies in $C^{0,2(1-\frac{1}{p})}$, hence recovering a result of Rupflin [Rup08] in the case of two-dimensional domains. Rivière has informed us that our regularity assertion in the particular case $f \equiv 0$ will also be made in the final version of [Riv09], based on a different proof.

We remark that the estimate (4.3) fails without the smallness of ω hypothesis. More precisely, there exist a sequence $\omega_k \in L^2(B_1, so(m) \otimes \wedge^1 \mathrm{T}^* \mathbb{R}^2)$ uniformly bounded in L^2, and a sequence of weak solutions $u_k \in W^{1,2}(B_1, \mathbb{R}^m)$ to the equation

$$-\Delta u_k = \omega_k.\nabla u_k,$$

uniformly bounded in $W^{1,2}$, such that u_k is unbounded in any $W^{2,p}$ space with $p \in (1,2)$. (A sequence of harmonic maps undergoing bubbling would provide an example, for instance the sequence given in section 5.6.)

Estimate (4.3) implies that for any sequence $\omega_k \in L^2(B_1, so(m) \otimes \wedge^1 \mathrm{T}^* \mathbb{R}^2)$ with $\|\omega_k\|_{L^2(B_1)} \leq$

η_0, and any sequence of weak solutions $u_k \in W^{1,2}(B_1, \mathbb{R}^m)$ to the equation

$$-\Delta u_k = \omega_k . \nabla u_k + f_k,$$

with u_k uniformly bounded in $W^{1,2}$ and f_k uniformly bounded in some space L^p for $p \in (1,2)$, we may deduce that u_k is locally uniformly bounded in $W^{2,p}$. By the theorem of Rellich-Kondrachov, we can deduce that u_k is precompact in $W^{1,t}(B_{1/2})$ for any $t < \frac{2p}{2-p}$.

In chapter 5, we work somewhat harder to prove a stronger compactness result, extending a recent theorem of Li and Zhu [LZ09], in which we assume merely that the inhomogeneous terms f_k are bounded in $L \log L$ (a space larger than any of the L^p spaces with $p > 1$, but slightly smaller than L^1).

At the heart of these results is a collection of energy/decay estimates which we summarise in the following theorem.

Theorem 4.2.2 (Main supporting theorem)**.** *Suppose $u \in W^{1,2}(B_1, \mathbb{R}^m)$ is a weak solution to*

$$-\Delta u = \omega . \nabla u + f, \qquad f \in L \log L(B_1, \mathbb{R}^m) \tag{4.4}$$

on the unit disc in $\mathbb{R}^2$, where $\omega \in L^2(B_1, so(m) \otimes \wedge^1 \mathrm{T}^* \mathbb{R}^2)$. Writing $\bar{u} = \fint_{B_1} u$,

1. *there exist $\eta = \eta(m) > 0$ and $K_1 = K_1(m) < \infty$ such that if $\|\omega\|_{L^2(B_1)} \leq \eta$, then for all $r \in (0, 1/2]$ we have*

$$\|\nabla u\|^2_{L^2(B_r)} \leq K_1 \left(\|\omega\|^2_{L^2(B_1)} \|\nabla u\|^2_{L^2(B_1)} + r^2 \|u - \bar{u}\|^2_{L^1(B_1)} + \|f\|_{L^1(B_1)} \|f\|_{L \log L(B_1)} \right) \tag{4.5}$$

   and

$$\|\nabla^2 u\|_{L^1(B_r)} \leq K_1 \left(\|\omega\|_{L^2(B_1)} \|\nabla u\|_{L^2(B_1)} + r^2 \|u - \bar{u}\|_{L^1(B_1)} + \|f\|_{L \log L(B_1)} \right); \tag{4.6}$$

2. *for all $\delta > 0$ there exist $\eta = \eta(m, \delta) > 0$ and $K_2 = K_2(m, \delta) < \infty$ such that if $\|\omega\|_{L^2(B_1)} \leq \eta$, then for all $r \in (0, 1]$ we have*

$$\begin{aligned} \|\nabla u\|^2_{L^2(B_r)} &\leq (1+\delta) r^2 \|\nabla u\|^2_{L^2(B_1)} + \\ &+ K_2 \left(\|\omega\|^2_{L^2(B_1)} \|\nabla u\|^2_{L^2(B_1)} + \|f\|_{L^1(B_1)} \|f\|_{L \log L(B_1)} \right). \end{aligned} \tag{4.7}$$

Although we will not need it in this work, we note that the first part of the theorem will also yield estimates for ∇u in the Lorentz space $L^{2,1}$ (by the embedding $W^{1,1} \hookrightarrow L^{2,1}$).

The estimates of the first part of the theorem are interior estimates which have the weakest norms of u on the right-hand side. By combining them with a standard covering argument, we will also derive the following optimal global estimate:

Theorem 4.2.3. *With u and f as in Theorem 4.2.2 and $U \subset\subset B_1$ there exist an $\eta_1 = \eta_1(m) > 0$ and $C = C(m,U) < \infty$ such that if $\|\omega\|_{L^2(B_1)} \le \eta_1$, then*

$$\|u\|_{W^{2,1}(U)} \le C\left(\|u\|_{L^1(B_1)} + \|f\|_{L\log L(B_1)}\right).$$

The second part of Theorem 4.2.2 is used to obtain both the regularity result Theorem 4.2.1 and the compactness result Theorem 5.2.1.

We remark that Theorems 4.2.1, 5.2.1, 4.2.2 and 4.2.3 all fail if we drop the antisymmetry hypothesis on ω.

4.3 Optimal regularity

We will now exhibit examples of solutions to (4.1) which are in $W^{2,p}_{loc}$ for all $p < 2$ but no better, showing that the regularity given in Theorem 4.2.1 is the best possible in this generality.

To make the computations easier we first remark that the PDE (4.1) is conformally invariant in particular suppose that $u \in W^{1,2}_{loc}(U,\mathbb{R}^m)$ weakly solves (4.1) on some domain $U \subset \mathbb{R}^2$ with $\|\nabla u\|_{L^2(U)} \le C < \infty$ and some $\omega \in L^2(U, so(m) \otimes \wedge^1 \mathrm{T}^*\mathbb{R}^2)$. Now let $\phi : V \to U$ be a conformal diffeomorphism and set $\tilde{u} = u \circ \phi$ and $\tilde{\omega} = \phi^*\omega$. It should be clear that these objects solve

$$-\Delta_g \tilde{u} = \langle \tilde{\omega}, \mathrm{d}\tilde{u}\rangle_g$$

where $g = \phi^*\delta$ and δ is the Euclidean metric. We leave it as an exercise to check that

$$\|\tilde{\omega}\|_{L^2(V)} = \|\omega\|_{L^2(U)},$$

$$\|\mathrm{d}\tilde{u}\|_{L^2(V)} = \|\mathrm{d}u\|_{L^2(U)},$$

and

$$-\Delta \tilde{u} = \langle \tilde{\omega}, \mathrm{d}\tilde{u}\rangle_\delta.$$

Now we consider $u : B_{e^{-1}} \to \mathbb{R}^2$ given by $u(x,y) = (x,y)t$ where $t = -\log r$ and $r = (x^2+y^2)^{\frac{1}{2}}$. It can be checked that $u \in W^{2,p}(B_{e^{-1}})$ for any $p < 2$ with $\nabla^2 u \notin L^2_{loc}(B_{e^{-1}})$. In order to simplify the calculations we work on the cylinder $S^1 \times (1,\infty)$ via the conformal diffeomorphism $\phi : S^1 \times (1,\infty) \to B_{e^{-1}} \backslash \{0\}$ where $\phi(\theta,t) = e^{-t}(\cos\theta, \sin\theta)$. We see that

$\tilde{u}(\theta, t) = te^{-t}(\cos\theta, \sin\theta)$, and defining

$$\tilde{\omega} = \frac{2}{t}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathrm{d}\theta = \tilde{\omega}^{\theta}\mathrm{d}\theta$$

we can check that

$$-\Delta\tilde{u} = 2e^{-t}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

and

$$\tilde{\omega}.\nabla\tilde{u} = \tilde{\omega}^{\theta}\,\tilde{u}_{\theta} = \frac{2}{t}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} te^{-t}\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = 2e^{-t}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

where the inner product and the Laplacian are with respect to the metric $dt^2 + d\theta^2$.

Converting everything back into (x, y)-coordinates in $B_{e^{-1}}$ we see that

$$-\Delta u = \frac{2}{r^2}\begin{pmatrix} x \\ y \end{pmatrix} \in L^{2,\infty},$$

showing in fact that $u \in W^{2,p}_{loc}$ for all $p < 2$. Also we can check that

$$\omega = \frac{2}{\log\left(\frac{1}{r}\right) r^2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(-ydx + xdy)$$

with

$$\nabla u = \begin{pmatrix} \log\left(\frac{1}{r}\right) - \frac{x^2}{r^2} \\ -\frac{xy}{r^2} \end{pmatrix} dx + \begin{pmatrix} -\frac{xy}{r^2} \\ \log\left(\frac{1}{r}\right) - \frac{y^2}{r^2} \end{pmatrix} dy.$$

We leave it to the reader to check $\omega \in L^2$. Actually we should note here that this example lies in $W^{2,(2,\infty)}$ and we also have $\nabla u \in BMO$ (this can be checked directly or by using $W^{1,(n,\infty)}(\mathbb{R}^n) \hookrightarrow BMO(\mathbb{R}^n)$.) In other words it looks like Theorem 4.2.1 could be improved. We show however, that this is not the case.

Consider now $u : B_{e^{-1}} \to \mathbb{R}^2$ given by $u(x, y) = (x, y)t^2$, we leave it to the reader to check that u solves (4.1) for

$$\omega = \frac{2(1 + 2\log r)}{(r\log r)^2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(-ydx + xdy) \in L^2$$

(again it is easier to convert everything into (θ, t)-coordinates first to see how this works, or check using the formulae below). Also we have

$$\Delta u = \frac{2(2\log r + 1)}{r^2}\begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$\nabla u = \begin{pmatrix} (\log r)^2 + \frac{2x^2\log r}{r^2} \\ \frac{2xy\log r}{r^2} \end{pmatrix} dx + \begin{pmatrix} \frac{2xy\log r}{r^2} \\ (\log r)^2 + \frac{2y^2\log r}{r^2} \end{pmatrix} dy.$$

Thus we see that $u \in W^{2,p}_{loc}$ for all $p < 2$ with $\nabla u \in L^q_{loc}$ for any $q < \infty$, but $\nabla u \notin BMO$ thus showing that we have $u \notin W^{2,(2,\infty)}_{loc}$ (by $W^{1,(n,\infty)}(\mathbb{R}^n) \hookrightarrow BMO(\mathbb{R}^n)$).

4.4 Proof of the decay estimates, Theorem 4.2.2

Most of the work in the proof will be common to both parts of the theorem. We will be referring to the δ of the second part with the understanding that in the case of the first part, we could just set $\delta = 1$.

We start off with $\eta = \epsilon$, taken from Theorem 3.2.5, and will assume throughout that $\|\omega\|_{L^2(B_1)} \leq \eta$, with the understanding that the upper bound η will be lowered at finitely many stages during the proof. For our weak solution u to (4.4) corresponding to ω, we will assume, without loss of generality, that $\bar{u} = \frac{1}{|B_1|}\int_{B_1} u = 0$.

To begin with we use Rivière's decomposition of ω (Theorem 3.2.5) in order to rewrite the equation (4.4) (equations (4.10) and (4.11) below). Theorem 3.2.5 gives us $A \in W^{1,2}(B_1, GL_m(\mathbb{R})) \cap L^\infty(B_1, GL_m(\mathbb{R}))$, $B \in W^{1,2}_0(B_1, gl_m(\mathbb{R}))$ and $C = C(m) < \infty$ so that

$$\nabla A - A\omega = \nabla^\perp B \tag{4.8}$$

and

$$\|\nabla A\|_{L^2(B_1)} + \|\nabla B\|_{L^2(B_1)} + \|\mathrm{dist(A,SO(m)}\|_{\mathrm{L^\infty(B_1)}} \leq \mathrm{C}\|\omega\|_{\mathrm{L^2(B_1)}}. \tag{4.9}$$

(We have actually replaced A by A^{-1} here, compared to how it appears in the statement of Theorem 3.2.5.)

Now by (4.8) we have

$$\begin{aligned} \mathrm{div}(A\nabla u) &= \nabla A.\nabla u + A\Delta u \\ &= \nabla A.\nabla u - A\omega.\nabla u - Af \\ &= \nabla^\perp B.\nabla u - Af \end{aligned} \tag{4.10}$$

and trivially

$$\operatorname{curl}(A\nabla u) = \nabla^{\perp}A.\nabla u. \tag{4.11}$$

We note here that the above equations only hold in a weak sense, and more care should be taken in their calculation. We illustrate this for (4.10): *A priori* $\operatorname{div}(A\nabla u)$ is a distribution, so for $\phi \in C_c^{\infty}(B_1)$ we have

$$\begin{aligned}
\operatorname{div}(A\nabla u)[\phi] &= -\int_{B_1} A\nabla u.\nabla\phi \\
&= \int_{B_1} (\nabla A.\nabla u)\phi - \nabla(\phi A).\nabla u \\
&= \int_{B_1} (\nabla A.\nabla u)\phi - (A\omega.\nabla u)\phi - Af\phi \qquad \text{since } u \text{ weakly solves (4.4)} \\
&= \int_{B_1} (\nabla^{\perp}B.\nabla u - Af)\phi = (\nabla^{\perp}B.\nabla u - Af)[\phi].
\end{aligned}$$

We will now essentially carry out a Hodge decomposition of $A\nabla u$ in B_1 using the expressions (4.10) and (4.11). We first extend all the quantities arising above to functions on $\mathbb{R}^2$.

Let $Ex : W^{1,2}(B_1) \to W^{1,2}(\mathbb{R}^2) \cap W_0^{1,2}(B_2)$ be a bounded extension operator. Denote $\tilde{u} = Ex(u) \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^m)$ and note that since we are assuming $\int_{B_1} u = 0$, by the Poincaré inequality and by standard properties of Ex we have

$$\|\tilde{u}\|_{W^{1,2}(\mathbb{R}^2)} \le C\|u\|_{W^{1,2}(B_1)} \le C\|\nabla u\|_{L^2(B_1)}$$

and $u = \tilde{u}$ in B_1.

For A, first let $\hat{A} = A - \frac{1}{|B_1|}\int_{B_1} A$ and $\tilde{A} = Ex(\hat{A}) \in W^{1,2}(\mathbb{R}^2, gl_m(\mathbb{R}))$. Noting that $\int_{B_1} \hat{A} = 0$ and using the same argument as for u we have

$$\|\tilde{A}\|_{W^{1,2}(\mathbb{R}^2)} \le C\|\nabla A\|_{L^2(B_1)}$$

here we have used that $\nabla\tilde{A} = \nabla\hat{A} = \nabla A$ in B_1. Notice also that $\tilde{A}\nabla\tilde{u} + \left(\frac{1}{|B_1|}\int_{B_1} A\right)\nabla\tilde{u} = A\nabla u$ in B_1.

We extend f and B by zero (without relabelling), so by Remark 2.7.6, $f \in h^1(\mathbb{R}^2)$ with $\|f\|_{h^1(\mathbb{R}^2)} \le C\|f\|_{L\log L(B_1)}$.

Now we define

$$D := N[\nabla^{\perp}B.\nabla\tilde{u}],$$

$$E := N[\nabla^{\perp}\tilde{A}.\nabla\tilde{u}],$$

$$F := -N[Af],$$

where N is the Newtonian potential (see section 2.2.3). Note that the quantity Af is well defined on the whole of $\mathbb{R}^2$ by the definition of f. Finally let

$$H := \tilde{A}\nabla\tilde{u} + \left(\frac{1}{|B_1|}\int_{B_1} A\right)\nabla\tilde{u} - \nabla D - \nabla F - \nabla^\perp E.$$

The first thing to notice about H is that

$$H = A\nabla u - \nabla D - \nabla F - \nabla^\perp E \tag{4.12}$$

in B_1. Hence we have

$$\operatorname{div}(H) = \operatorname{div}(A\nabla u) - \Delta(D+F) = \operatorname{div}(A\nabla u) - \nabla^\perp B.\nabla\tilde{u} + Af = 0$$

weakly in B_1, and a similar calculation shows $\operatorname{curl}(H) = 0$ weakly in B_1. (Again care must be taken in checking these.) Therefore H is harmonic in B_1 (i.e. corresponds to a harmonic 1-form).

Suppose $r \in (0,1]$. (For some estimates later it will need to be less than $\frac{1}{2}$.)

Without loss of generality, we may assume that $\delta \in (0,1]$. (Recall that when addressing the first part of the theorem, we are just setting $\delta = 1$.) For η small enough, depending on δ, we may assume (by (4.9)) that A is close to a special-orthogonal matrix in the sense that both A and A^{-1} change the length of any vector by at most a factor of $1+\delta$. Therefore

$$\begin{aligned}
\|\nabla u\|^2_{L^2(B_r)} &\le (1+\delta)^2\|A\nabla u\|^2_{L^2(B_r)} \\
&\le (1+3\delta)\|A\nabla u\|^2_{L^2(B_r)} \\
&\le (1+4\delta)\|H\|^2_{L^2(B_r)} + C(\|\nabla D\|^2_{L^2(B_r)} + \|\nabla F\|^2_{L^2(B_r)} + \|\nabla E\|^2_{L^2(B_r)}),
\end{aligned} \tag{4.13}$$

where C is dependent on δ. In order to obtain the inequalities of Theorem 4.2.2 we estimate $\|H\|_{L^2(B_r)}$, $\|\nabla D\|_{L^2(B_r)}$, $\|\nabla F\|_{L^2(B_r)}$ and $\|\nabla E\|_{L^2(B_r)}$.

First we consider $\nabla D = \nabla N[\nabla^\perp B.\nabla\tilde{u}]$ and $\nabla E = \nabla N[\nabla^\perp\tilde{A}.\nabla\tilde{u}]$. Notice that by the work of Coifman-Lions-Meyer-Semmes [CLMS93] and the fact that

$$\nabla N : \mathscr{H}^1(\mathbb{R}^2) \to L^{2,1}(B_1)$$

is a bounded linear operator (see Remark 2.7.4) we have,

$$\begin{aligned}
\|\nabla D\|_{L^2(B_1)}+\|\nabla E\|_{L^2(B_1)} &\leq C\left(\|\nabla D\|_{L^{2,1}(B_1)}+\|\nabla E\|_{L^{2,1}(B_1)}\right)\\
&= C\left(\|\nabla N[\nabla^{\perp}B.\nabla\tilde{u}]\|_{L^{2,1}(B_1)}+\|\nabla N[\nabla^{\perp}\tilde{A}.\nabla\tilde{u}]\|_{L^{2,1}(B_1)}\right)\\
&\leq C\left(\|\nabla^{\perp}B.\nabla\tilde{u}\|_{\mathcal{H}^1(\mathbb{R}^2)}+\|\nabla^{\perp}\tilde{A}.\nabla\tilde{u}\|_{\mathcal{H}^1(\mathbb{R}^2)}\right)\\
&\leq C\left(\|\nabla B\|_{L^2(B_1)}+\|\nabla A\|_{L^2(B_1)}\right)\|\nabla u\|_{L^2(B_1)}\\
&\leq C\|\omega\|_{L^2(B_1)}\|\nabla u\|_{L^2(B_1)} \qquad (4.14)
\end{aligned}$$

where we have also used the continuous embedding $L^{2,1}\hookrightarrow L^2$ and the estimate from Theorem 3.2.5.

For $\nabla F=-\nabla N[Af]$ we use (Theorem 2.2.1) that the Riesz potential $\nabla N: L^1(B_1)\to L^{2,\infty}(B_1)$ is a bounded operator; also $\nabla N: h^1(\mathbb{R}^2)\to L^{2,1}(B_1)$ is bounded (Remark 2.7.4). We will also use the following: $L^{2,\infty}$ is the dual of $L^{2,1}$; if $f\in L\log L(B_1)$ then for any $g\in L^{\infty}$, $gf\in L\log L(B_1)$ and $\|gf\|_{L\log L(B_1)}\leq\|g\|_{L^{\infty}}\|f\|_{L\log L(B_1)}$ and finally we use the continuous embedding $L\log L(B_1)\hookrightarrow h^1(\mathbb{R}^2)$. We have

$$\begin{aligned}
\|\nabla F\|^2_{L^2(B_1)} &\leq C\|\nabla F\|_{L^{2,\infty}(B_1)}\|\nabla F\|_{L^{2,1}(B_1)}\\
&\leq C\|Af\|_{L^1(B_1)}\|Af\|_{h^1(\mathbb{R}^2)}\\
&\leq C\|f\|_{L^1(B_1)}\|Af\|_{L\log L(B_1)}\\
&\leq C\|f\|_{L^1(B_1)}\|f\|_{L\log L(B_1)}. \qquad (4.15)
\end{aligned}$$

Also, using merely the boundedness of $\nabla N: L^1(B_1)\to L^{2,\infty}(B_1)\hookrightarrow L^1(B_1)$, we have

$$\|\nabla F\|_{L^1(B_1)}\leq C\|f\|_{L^1(B_1)}. \qquad (4.16)$$

From here, we proceed differently in order to prove the two different parts of the theorem. For the first part, we now estimate $\|H\|_{L^1(B_{2/3})}$ and apply standard estimates for harmonic functions in order to estimate $\|H\|_{L^2(B_r)}$: Using estimates (4.14) and (4.16), we have

$$\begin{aligned}
\|H\|_{L^1(B_{2/3})} &\leq C\left(\|\nabla u\|_{L^1(B_{2/3})}+\|\nabla D\|_{L^1(B_{2/3})}+\|\nabla E\|_{L^1(B_{2/3})}+\|\nabla F\|_{L^1(B_{2/3})}\right)\\
&\leq C\left(\|u\|_{L^1(B_1)}+\|f\|_{L^1(B_1)}+\|\omega\|_{L^2(B_1)}\|\nabla u\|_{L^2(B_1)}\right) \qquad (4.17)
\end{aligned}$$

where we have also used the estimate

$$\|\nabla u\|_{L^1(B_{2/3})}\leq C(\||u\|_{L^1(B_1)}+\|\omega\|_{L^2(B_1)}\|\nabla u\|_{L^2(B_1)})$$

which follows from Propositions 2.3.1 and 2.4.2.

Since H is harmonic we have pointwise estimates on H and its derivatives on the interior of $B_{2/3}$ in terms of $\|H\|_{L^1(B_{2/3})}$, and in particular

$$\begin{aligned}\|H\|_{L^\infty(B_{1/2})} + \|\nabla H\|_{L^\infty(B_{1/2})} &\leq C\|H\|_{L^1(B_{2/3})} \\ &\leq C(\|u\|_{L^1(B_1)} + \|f\|_{L^1(B_1)} + \|\omega\|_{L^2(B_1)}\|\nabla u\|_{L^2(B_1)})\end{aligned}$$

by (4.17). Therefore if we consider $r \in (0, \frac{1}{2}]$, then

$$\begin{aligned}\|H\|^2_{L^2(B_r)} &\leq \pi r^2\|H\|^2_{L^\infty(B_r)} \\ &\leq Cr^2\left(\|u\|^2_{L^1(B_1)} + \|f\|^2_{L^1(B_1)} + \|\omega\|^2_{L^2(B_1)}\|\nabla u\|^2_{L^2(B_1)}\right),\end{aligned} \tag{4.18}$$

and

$$\begin{aligned}\|\nabla H\|_{L^1(B_r)} &\leq \pi r^2\|\nabla H\|_{L^\infty(B_r)} \\ &\leq Cr^2\left(\|u\|_{L^1(B_1)} + \|f\|_{L^1(B_1)} + \|\omega\|_{L^2(B_1)}\|\nabla u\|_{L^2(B_1)}\right).\end{aligned} \tag{4.19}$$

Now, looking back at inequality (4.13) and using (4.14), (4.15) and (4.18) we have

$$\|\nabla u\|^2_{L^2(B_r)} \leq C\left(\|\omega\|^2_{L^2(B_1)}\|\nabla u\|^2_{L^2(B_1)} + r^2\|u\|^2_{L^1(B_1)} + \|f\|_{L^1(B_1)}\|f\|_{L\log L(B_1)}\right)$$

which is the first inequality that we seek from the first part of the theorem.

In order to get the second estimate (4.6) of the first part of the theorem, we return to the Hodge decomposition (4.12) which tells us that

$$\nabla u = A^{-1}(H + \nabla D + \nabla F + \nabla^\perp E)$$

in B_1, and therefore

$$\nabla^2 u = \nabla A^{-1}.(H + \nabla D + \nabla F + \nabla^\perp E) + A^{-1}(\nabla H + \nabla^2 D + \nabla^2 F + \nabla\nabla^\perp E),$$

with

$$\begin{aligned}\|\nabla^2 u\|_{L^1(B_r)} &\leq \|\nabla A^{-1}.(H + \nabla D + \nabla F + \nabla^\perp E)\|_{L^1(B_r)} \quad (=\mathrm{I}) \\ &+ \|A^{-1}(\nabla H + \nabla^2 D + \nabla^2 F + \nabla\nabla^\perp E)\|_{L^1(B_r)}. \quad (=\mathrm{II})\end{aligned} \tag{4.20}$$

Using (4.9), (4.14), (4.15) and (4.18) we have (assuming also $\eta < 1$)

$$\begin{aligned}\mathrm{I} &\leq \|\nabla A^{-1}\|_{L^2(B_r)}\|H+\nabla D+\nabla E+\nabla^{\perp}F\|_{L^2(B_r)}\\ &\leq C\left(r\|\omega\|_{L^2(B_1)}\|u\|_{L^1(B_1)}+\|\omega\|_{L^2(B_1)}\|\nabla u\|_{L^2(B_1)}+\|f\|_{L\log L(B_1)}\right).\end{aligned} \tag{4.21}$$

Now we use the fact that the operators $\nabla^2 N : h^1(\mathbb{R}^2) \to L^1(B_1)$ (Remark 2.6.15) and $\nabla^2 N : \mathscr{H}^1(\mathbb{R}^2) \to L^1(\mathbb{R}^2)$ (Theorem 2.6.4) are bounded and the estimates (4.9) and (4.19) to conclude

$$\begin{aligned}\mathrm{II} &\leq \|A^{-1}\|_{L^\infty(B_r)}\|\nabla H+\nabla^2 D+\nabla^2 F+\nabla\nabla^{\perp}E\|_{L^1(B_r)}\\ &\leq C\left(r^2\|u\|_{L^1(B_1)}+\|\omega\|_{L^2(B_1)}\|\nabla u\|_{L^2(B_1)}+\|f\|_{L\log L(B_1)}\right).\end{aligned} \tag{4.22}$$

Since we have assumed without loss of generality that $\int_{B_1} u = 0$, by an application of the Poincaré inequality and looking back at (4.20), (4.21) and (4.22) we have

$$\|\nabla^2 u\|_{L^1(B_r)} \leq C(\|\omega\|_{L^2(B_1)}\|\nabla u\|_{L^2(B_1)} + r^2\|u\|_{L^1(B_1)} + \|f\|_{L\log L(B_1)}),$$

as desired.

For the second part of the theorem, we return to a general $r \in (0,1]$. We will now control H using the standard decay estimate

$$\|H\|^2_{L^2(B_r)} \leq r^2\|H\|^2_{L^2(B_1)}$$

which holds since $|H|^2$ is subharmonic.

Then using (4.13) and (4.14) and (4.15) again, we find that

$$\begin{aligned}\|\nabla u\|^2_{L^2(B_r)} &\leq (1+4\delta)\|H\|^2_{L^2(B_r)} + C(\|\nabla D\|^2_{L^2(B_r)} + \|\nabla F\|^2_{L^2(B_r)} + \|\nabla E\|^2_{L^2(B_r)})\\ &\leq (1+4\delta)r^2\|H\|^2_{L^2(B_1)} + C(\|\nabla D\|^2_{L^2(B_1)} + \|\nabla F\|^2_{L^2(B_1)} + \|\nabla E\|^2_{L^2(B_1)})\\ &\leq (1+5\delta)r^2\|A\nabla u\|^2_{L^2(B_1)} + C(\|\nabla D\|^2_{L^2(B_1)} + \|\nabla F\|^2_{L^2(B_1)} + \|\nabla E\|^2_{L^2(B_1)})\\ &\leq (1+5\delta)(1+\delta)^2 r^2\|\nabla u\|^2_{L^2(B_1)} + C(\|\nabla D\|^2_{L^2(B_1)} + \|\nabla F\|^2_{L^2(B_1)} + \|\nabla E\|^2_{L^2(B_1)})\\ &\leq (1+100\delta)r^2\|\nabla u\|^2_{L^2(B_1)} + C\left(\|\omega\|^2_{L^2(B_1)}\|\nabla u\|^2_{L^2(B_1)} + \|f\|_{L^1(B_1)}\|f\|_{L\log L(B_1)}\right).\end{aligned}$$

Thus, by repeating the argument with δ reduced by a factor of 100, we conclude the proof.

4.5 Proof of the $W^{2,1}$ estimate, Theorem 4.2.3

We can say immediately that the η_1 whose existence is claimed in the theorem can be chosen as $\eta_1^2 = \min\left\{\eta^2, \frac{\epsilon_0}{K_1}\right\}$, where ϵ_0 is that given in Lemma C.0.5 corresponding to $k = 4$, and K_1 and η are from the first part of Theorem 4.2.2.

We would like to rescale the first estimate (4.5) of the first part of Theorem 4.2.2, in the case that $r = \frac{1}{2}$. Indeed, adopting the notation of appendix B, we know that

$$\begin{aligned}\|\nabla\hat{u}\|^2_{L^2(B_{1/2})} &\le K_1\left(\|\hat{\omega}\|^2_{L^2(B_1)}\|\nabla\hat{u}\|^2_{L^2(B_1)} + \|\hat{u}\|^2_{L^1(B_1)} + \|\hat{f}\|_{L^1(B_1)}\|\hat{f}\|_{L\log L(B_1)}\right)\\ &\le K_1\|\hat{\omega}\|^2_{L^2(B_1)}\|\nabla\hat{u}\|^2_{L^2(B_1)} + C\left(\|\hat{u}\|^2_{L^1(B_1)} + \|\hat{f}\|^2_{L\log L(B_1)}\right)\end{aligned}$$

and (again by appendix B) this translates to

$$\|\nabla u\|^2_{L^2(B_{R/2}(x_0))} \le K_1\|\omega\|^2_{L^2(B_R(x_0))}\|\nabla u\|^2_{L^2(B_R(x_0))} + C\left(R^{-4}\|u\|^2_{L^1(B_R(x_0))} + \|f\|^2_{L\log L(B_R(x_0))}\right).$$

Using our upper bound for η and the fact that $R \le 1$ we have, in particular

$$\|\nabla u\|^2_{L^2(B_{R/2}(x_0))} \le \epsilon_0\|\nabla u\|^2_{L^2(B_R(x_0))} + CR^{-4}\left(\|u\|^2_{L^1(B_1)} + \|f\|^2_{L\log L(B_1)}\right).$$

Letting $\Gamma = C\left(\|u\|^2_{L^1(B_1)} + \|f\|^2_{L\log L(B_1)}\right)$ we are precisely in the set-up of Lemma C.0.5, since this estimate is true in particular for all $B_{2R}(x_0) \subset B_1$. Therefore

$$\|\nabla u\|_{L^2(B_{1/2})} \le C\left(\|u\|_{L^1(B_1)} + \|f\|_{L\log L(B_1)}\right). \tag{4.23}$$

It remains to improve this estimate to control the second derivatives, and for that we use the second estimate (4.6) of the first part of Theorem 4.2.2, in the case $r = \frac{1}{2}$, which we then scale by a factor $\frac{1}{2}$ to give:

$$\|\nabla^2 u\|_{L^1(B_{1/4})} \le C\left(\|\omega\|_{L^2(B_{1/2})}\|\nabla u\|_{L^2(B_{1/2})} + \|u\|_{L^1(B_{1/2})} + \|f\|_{L\log L(B_{1/2})}\right).$$

Combining with (4.23) then yields

$$\|\nabla^2 u\|_{L^1(B_{1/4})} \le C\left(\|u\|_{L^1(B_1)} + \|f\|_{L\log L(B_1)}\right),$$

and a simple rescaling and covering argument gives us that for any compactly contained $U \subset\subset B_1$ there is a $C = C(U, m) < \infty$ such that

$$\|u\|_{W^{2,1}(U)} \le C\left(\|u\|_{L^1(B_1)} + \|f\|_{L\log L(B_1)}\right).$$

4.6 Proof of the optimal regularity, Theorem 4.2.1

We will need an additional lemma, which expresses the decay of energy of solutions u. This Lemma also follows from the case $n = 2$ in Proposition 6.3.1. We prove this using the decay estimates in Theorem 4.2.2, the full regularity result then follows from the proof of Theorem 6.2.1.

Lemma 4.6.1. *Suppose $u \in W^{1,2}(B_1, \mathbb{R}^m)$ is a weak solution to*

$$-\Delta u = \omega.\nabla u + f$$

where $\omega \in L^2(B_1, so(m) \otimes \wedge^1 \mathrm{T}^*\mathbb{R}^2)$ and $f \in L^p$ for some $p \in (1,2)$. Then $\nabla u \in M^{2,4(1-\frac{1}{p})}_{loc}$.

Proof. We prove the theorem under the hypothesis that $\|\omega\|_{L^2(B_1)} \leq \eta_4$ (defined below), then the result follows from a simple covering argument.

For reasons that will become apparent, choose $\delta \in (0,1]$ sufficiently small so that

$$\lambda := \frac{(1+2\delta)}{4} < 2^{-4(1-1/p)} =: \Lambda \in \left(\frac{1}{4}, 1\right) \tag{4.24}$$

We can now choose $\eta_4 := \min\{\eta, \sqrt{\frac{\delta}{4K_2}}\}$, where η is from the second part of Theorem 4.2.2, depending on the δ we have just chosen and therefore on p (as well as m) and where K_2 is also from the second part of Theorem 4.2.2. Now take an arbitrary point $x_0 \in B_{1/2}$ and any $R \in (0,1/2)$. Estimate (4.7) in the case that $r = \frac{1}{2}$, applied to the rescaled quantities defined in appendix B yields

$$\begin{aligned}\|\nabla \hat{u}\|^2_{L^2(B_{1/2})} &\leq \frac{(1+\delta)}{4}\|\nabla \hat{u}\|^2_{L^2(B_1)} + K_2\left(\|\hat{\omega}\|^2_{L^2(B_1)}\|\nabla \hat{u}\|^2_{L^2(B_1)} + \|\hat{f}\|_{L^1(B_1)}\|\hat{f}\|_{L\log L(B_1)}\right)\\ &\leq \lambda\|\nabla \hat{u}\|^2_{L^2(B_1)} + C\|\hat{f}\|^2_{L^p(B_1)}.\end{aligned}$$

Reversing the scaling, using Appendix B, we find that

$$\begin{aligned}\|\nabla u\|^2_{L^2(B_{R/2}(x_0))} &\leq \lambda\|\nabla u\|^2_{L^2(B_R(x_0))} + CR^{4(1-1/p)}\|f\|^2_{L^p(B_R(x_0))}\\ &\leq \lambda\|\nabla u\|^2_{L^2(B_R(x_0))} + [C\|f\|^2_{L^p(B_1)}]R^{4(1-1/p)}.\end{aligned}$$

Now applying what we have proved for $R = 2^{-k}$, with $k \in \{1,2\ldots\}$ and using the abbreviation $a_k := \|\nabla u\|^2_{L^2(B_{2^{-k}}(x_0))}$, we find that

$$\begin{aligned}a_{k+1} &\leq \lambda a_k + K_3 2^{-4(1-1/p)k}\\ &= \lambda a_k + K_3\Lambda^k\end{aligned}$$

where K_3 is independent of x_0. This recursion relation can be solved to yield

$$a_{k+1} \leq \lambda^k a_1 + K_3 \Lambda \frac{(\Lambda^k - \lambda^k)}{\Lambda - \lambda},$$

and by (4.24), this simplifies to

$$\|\nabla u\|^2_{L^2(B_{2^{-k}}(x_0))} =: a_k \leq C\Lambda^k.$$

Thus, for $r \in (0, 1/2]$ we have

$$\|\nabla u\|^2_{L^2(B_r(x_0))} \leq C r^{4(1-1/p)},$$

and hence the lemma is proved. □

Chapter 5

Compactness properties

5.1 Introduction

Here we study compactness properties of solutions to (4.1) with $f \in L\log L$, and show that under a smallness condition we have strong $W^{1,2}$-compactness. We begin by providing an example to again show that the anti-symmetry condition is necessary for $W^{1,2}$ compactness and not just regularity. We also show that if we replace $L\log L$ with the related local Hardy space h^1 then this compactness fails. We shall see also that this comes down to the fact that in dimension two the embedding $L\log L \subset\subset H^{-1}$ is compact. We show using Orlicz space methods that this generalises sufficiently to all dimensions and also for a greater range of Zygmund spaces. This last fact comes down to finding compact embeddings in the critical Sobolev embedding setting mentioned in section 2.7.

5.2 Results

Theorem 5.2.1 (Compactness)**.** *Suppose that we have a sequence* $\{u_k\} \subset W^{1,2}(B_1,\mathbb{R}^m)$ *of weak solutions to*

$$-\Delta u_k = \omega_k.\nabla u_k + f_k$$

on the unit disc in $\mathbb{R}^2$*, where* $\{\omega_k\} \subset L^2(B_1, so(m)\otimes \wedge^1 \mathrm{T}^*\mathbb{R}^2)$ *and* $\{f_k\} \subset L\log L(B_1,\mathbb{R}^m)$*. Suppose also that there exists* $\Lambda < \infty$ *such that*

$$\|u_k\|_{L^1(B_1)} + \|f_k\|_{L\log L(B_1)} \le \Lambda.$$

Then there exist an $\eta_2 = \eta_2(m) > 0$ *and* $u \in W^{1,2}_{loc}(B_1, \mathbb{R}^m)$ *such that if* $\|\omega_k\|_{L^2(B_1)} \leq \eta_2$ *then after passing to a subsequence*

$$\lim_{n\to\infty} \|u_k - u\|_{W^{1,2}(B_{1/2})} = 0.$$

We will show in section 5.6 that this result fails if we replace $L\log L$ by the related local Hardy space h^1. In the special case that $\{\omega_k\}$ is a precompact set in L^1, and u_k is uniformly bounded in $W^{1,2}$, this result was proved recently by Li and Zhu [LZ09].

Remark 5.2.2. The compactness result is ruling out concentration of energy as is done in [LZ09] – i.e. concentration of $\|\nabla u_k\|^2_{L^2}$. In contrast, we do not rule out concentration of $\|\nabla u_k\|^2_{L^{2,1}}$ or of the corresponding second order quantity $\|\nabla^2 u_k\|_{L^1}$. However, it will follow from our estimates (and in particular, (4.6) below) that if these latter concentrations occur we must have f_k concentrating in $L\log L$.

Even in the classical case that $\omega \equiv 0$ there is a consequence of such compactness which may be worth remarking, although one which would follow from previously known theory.

Corollary 5.2.3. *On the ball in* $\mathbb{R}^2$*, the embedding*

$$L\log L(B_1) \hookrightarrow H^{-1}(B_1)$$

is compact, where H^{-1} *is the dual space of* $W^{1,2}_0$*.*

In contrast, the example given in Section 5.6 also serves to show that the embedding $h^1(\mathbb{R}^2) \hookrightarrow H^{-1}(\mathbb{R}^2)$ is *not* compact, which has been pointed out previously by Yuxin Ge [Ge98, Remark 4.4]. (Strictly speaking, the example given in Section 5.6 and that of [Ge98, Remark 4.4] show that the embedding $\mathscr{H}^1(\mathbb{R}^2) \hookrightarrow H^{-1}(\mathbb{R}^2)$ is not compact, but since $\mathscr{H}^1(\mathbb{R}^2) \hookrightarrow h^1(\mathbb{R}^2)$, this is sufficient - see Section 2.6.)

We generalise Corollary 5.2.3 with the following

Theorem 5.2.4. *Let* $U \subset \mathbb{R}^n$ *be a bounded set and* $\beta > 1 - \frac{1}{n}$*, then we have*

$$L\log^{\beta} L(U) \subset\subset (W^{1,n}_0(U))^*.$$

We have the continuous embedding

$$L\log^{1-\frac{1}{n}} L(U) \hookrightarrow (W^{1,n}_0(U))^*.$$

5.3 Necessity of anti-symmetry

We consider a sequence of functions $\{u_k\} \subset W^{1,2}(B_1, \mathbb{R}^m)$ such that $\|u_k\|_{W^{1,2}(B_1)} \leq C < \infty$ and each a weak solution of the following

$$-\Delta u_k = \omega_k . \nabla u_k.$$

It should be clear that this counter-example is the standard one given by Frehse, at the end of section 2.3. Here $\omega_k \in L^2(B_1, M_m(\mathbb{R}) \otimes \wedge^1 \mathrm{T}^* \mathbb{R}^2)$ with $\|\omega_k\|_{L^2(B_1)} \to 0$. We will show below that such a sequence exists and it fails to have a strongly convergent subsequence in $W^{1,2}$.

Consider first $m = 1$ and let $u_k(x, y) = u_k(r) = (\log(ke))^{1/2} \left(\log\log\left(\frac{ke}{r}\right) - \log\log(ke) \right)$. We have that $|\nabla u_k|^2 = u_k'(r)^2 = \frac{\log(ke)}{r^2 \left(\log\left(\frac{ke}{r}\right)\right)^2}$ and for $R \in [0, 1]$

$$\begin{aligned}
\|\nabla u_k\|^2_{L^2(B_R)} &= \int_{B_R} \frac{\log(ke)}{r^2 \left(\log\left(\frac{ke}{r}\right)\right)^2} \, dx \, dy \\
&= 2\pi \log(ke) \int_0^R \frac{1}{r \left(\log\left(\frac{ke}{r}\right)\right)^2} \, dr \\
&= 2\pi \log(ke) \left[\left(\log\left(\frac{ke}{r}\right) \right)^{-1} \right]_0^R \\
&= 2\pi \frac{\log(ke)}{\log\left(\frac{ke}{R}\right)}.
\end{aligned}$$

This tells us two things in particular. First we see that $\|\nabla u_k\|^2_{L^2(B_1)} = 2\pi$ for all k, hence $\|u_k\|_{W^{1,2}(B_1)} \leq C < \infty$ (u has zero boundary values). Secondly we see that

$$\lim_{R \downarrow 0} \limsup_{k \to \infty} \|\nabla u_k\|_{L^2(B_R)} = 2\pi.$$

Thus we cannot expect any strong $W^{1,2}$-convergence.

Another simple calculation shows that

$$-\Delta u_k = \frac{\log(ke)^{1/2}}{r^2 \log\left(\frac{ke}{r}\right)^2}$$

and that $-\Delta u_k \in L^1(B_1)$.

Now we set

$$\omega_k := -\frac{1}{r^2 \log\left(\frac{ke}{r}\right)} (x\mathrm{d}x + y\mathrm{d}y)$$

and we see that $\|\omega_k\|^2_{L^2(B_1)} = \frac{2\pi}{\log(ke)} \to 0$ as $k \to \infty$.

Observing that

$$\nabla u_k = -\frac{(\log(ke))^{1/2}}{r \log\left(\frac{ke}{r}\right)} \left(\frac{x}{r}, \frac{y}{r}\right)$$

we see that $\omega_k . \nabla u_k = -\Delta u_k$.

Finally we see that $\{u_k\}$ satisfies what we want, and also it is easily checked that $u_k \to 0$ almost everywhere but $\|\nabla u_k\|^2_{L^2(B_1)} = 2\pi$ for all k so we cannot have strong $W^{1,2}$ convergence of any subsequence. Actually we see that $\|-\Delta u_k\|_{L^1(B_1)} \to 0$ which implies that we get strong $W^{1,q}$-convergence to zero for our sequence (for all $q < 2$).

For $m > 1$ consider $v_k : B_1 \to \mathbb{R}^m$ to have all m coordinate functions equal to u_k and let $\tilde{\omega}_k = Id_m \omega_k$.

5.4 Proof of the compactness, Theorem 5.2.1

Here we pick $\eta_2 = \min\{\eta_1, \eta, \sqrt{\frac{1}{2K_2}}\}$ where η_1 is from Theorem 4.2.3, and η and K_2 are from the second part of Theorem 4.2.2 for the choice $\delta = 1$. We know (by Theorem 4.2.3) that for all $U \subset\subset B_1$, our sequence $\{u_k\}$ is uniformly bounded in $W^{2,1}(U)$, so by the Sobolev embedding theorem there exists some $u \in W^{1,2}_{loc}(B_1)$ such that (up to a subsequence) $u_k \rightharpoonup u$ weakly in $W^{1,2}(B_{2/3})$. We also know that $\{\nabla u_k\}$ is uniformly bounded in $W^{1,1}(B_{2/3})$, so by Lemma D.0.7 (with $\nabla u_k = V_n$) if we have

$$\lim_{r \downarrow 0} \limsup_{k \to \infty} \|\nabla u_k\|_{L^2(B_r(x))} = 0 \tag{5.1}$$

for all $x \in B_{2/3}$, then

$$\nabla u_k \to \nabla u$$

strongly in $L^2_{loc}(B_{2/3})$ which would prove the theorem. Therefore, it remains to prove (5.1).

Now pick $x_0 \in B_{2/3}$ and $R \in (0, 1/2]$ small enough such that $B_R(x_0) \subset B_{2/3}$. Applying the second part of Theorem 4.2.2 to the rescaled scenario from appendix B (for each k) yields (for $r \in (0,1]$)

$$\|\nabla \hat{u}_k\|^2_{L^2(B_r)} \leq K_2 \|\hat{\omega}_k\|^2_{L^2(B_1)} \|\nabla \hat{u}_k\|^2_{L^2(B_1)} + 2r^2 \|\nabla \hat{u}_k\|^2_{L^2(B_1)} + K_2 \|\hat{f}_k\|_{L^1(B_1)} \|\hat{f}_k\|_{L\log L(B_1)}$$

and reversing the scaling leaves us with

$$\begin{aligned}\|\nabla u_k\|^2_{L^2(B_{rR}(x_0))} &\leq K_2\|\omega_k\|^2_{L^2(B_R(x_0))}\|\nabla u_k\|^2_{L^2(B_R(x_0))}\\ &\quad + 2r^2\|\nabla u_k\|^2_{L^2(B_R(x_0))} + K_2\|f_k\|_{L^1(B_R(x_0))}\|f_k\|_{L\log L(B_R(x_0))}\\ &\leq \frac{1}{2}\|\nabla u_k\|^2_{L^2(B_R(x_0))}\\ &\quad + C\left(r^2\|\nabla u_k\|^2_{L^2(B_R(x_0))} + \left[\log\left(\frac{1}{R}\right)\right]^{-1}\|f_k\|^2_{L\log L(B_R(x_0))}\right)\end{aligned}$$

using Lemma B.0.3.

Now, using that $\{u_k\}$ is uniformly bounded in $W^{2,1}(B_{2/3})$ and the hypotheses of the theorem, we have

$$\|\nabla u_k\|^2_{L^2(B_{rR}(x_0))} \leq 1/2\|\nabla u_k\|^2_{L^2(B_R(x_0))} + C\left(r^2 + \left[\log\left(\frac{1}{R}\right)\right]^{-1}\right)$$

Hence

$$\lim_{R\downarrow 0}\lim_{r\downarrow 0}\limsup_{k\to\infty}\|\nabla u_k\|^2_{L^2(B_{rR}(x_0))} \leq 1/2\lim_{R\downarrow 0}\lim_{r\downarrow 0}\limsup_{k\to\infty}\|\nabla u_k\|^2_{L^2(B_R(x_0))}$$

and we have shown that

$$\lim_{r\downarrow 0}\limsup_{k\to\infty}\|\nabla u_k\|_{L^2(B_r(x_0))} = 0$$

which proves the theorem.

5.5 Proof of the compact embedding $L\log L \hookrightarrow H^{-1}$, Corollary 5.2.3

In this section we use $\mathscr{H}^1 - BMO$ duality (see section 2.6 and [FS72]), the compactness result Theorem 5.2.1 and the continuous embedding $W^{1,2}(\mathbb{R}^2) \hookrightarrow BMO(\mathbb{R}^2)$ (Lemma 2.7.3) to prove the compactness of the embedding $L\log L(B_1) \hookrightarrow H^{-1}(B_1)$.

First we check that the embedding $L\log L \hookrightarrow H^{-1}$ exists and is continuous. We will realise $f \in L\log L(B_1)$ as a bounded linear functional on $W^{1,2}_0(B_1)$.

Recall from Remark 2.7.6 that if $f \in L\log L(B_1)$ then $f - \bar{f} \in \mathscr{H}^1(\mathbb{R}^2)$ and $\|f-\bar{f}\|_{\mathscr{H}^1(\mathbb{R}^2)} \leq$

$C\|f\|_{L\log L(B_1)}$. For $\phi \in W_0^{1,2}(B_1)$ we extend it by zero and calculate

$$\begin{aligned}\int_{B_1} f\phi &= \int (f-\bar{f})\phi + \int \bar{f}\phi \\ &\leq C\|f-\bar{f}\|_{\mathcal{H}^1(\mathbb{R}^m)}\|\phi\|_{BMO(\mathbb{R}^2)} + \frac{1}{\pi}\|f\|_{L^1(B_1)}\|\phi\|_{L^1(B_1)} \\ &\leq C\|f\|_{L\log L(B_1)}\|\phi\|_{W_0^{1,2}(B_1)}.\end{aligned}$$

Thus $f \in H^{-1}(B_1)$ and $\|f\|_{H^{-1}(B_1)} \leq C\|f\|_{L\log L(B_1)}$.

Now consider a sequence $\{f_k\} \subset L\log L(B_1)$ such that $\|f_k\|_{L\log L(B_1)} \leq \Lambda < \infty$. We can extend each f_k to be zero outside B_1 and consider the sequence of solutions $\{u_k\} \subset W_0^{1,2}(B_2)$ weakly solving

$$-\Delta u_k = f_k \qquad \text{on } B_2.$$

By the compactness of Theorem 5.2.1 for $\omega_k \equiv 0$ we can conclude that there exists some $u \in W^{1,2}(B_2)$ such that (up to a subsequence) $u_k \to u$ strongly in $W^{1,2}(B_1)$.

Writing $f = -\Delta u$ (which can clearly be viewed as an element of $H^{-1}(B_1)$) we see that

$$\begin{aligned}\|f_k - f\|_{H^{-1}(B_1)} &= \sup_{\phi\in W_0^{1,2}(B_1)\ \|\phi\|_{W_0^{1,2}(B_1)}=1} \int (f_k - f)\phi \\ &= \sup_{\phi\in W_0^{1,2}(B_1)\ \|\phi\|_{W_0^{1,2}(B_1)}=1} \int (\nabla u_k - \nabla u).\nabla\phi \\ &\leq \|\nabla u_k - \nabla u\|_{L^2(B_1)} \to 0\end{aligned}$$

as $k \to \infty$.

5.6 $L\log L$ cannot be replaced by h^1

Here we present a counterexample to the compactness Theorem 5.2.1 when we allow $f_k \in h^1(B_1)$. Our example will have $\omega_k \equiv 0$ for all k and $u_k : B_1 \to \mathbb{S}^2$ will be a sequence of harmonic maps with bounded energy that undergoes bubbling.

Let $\pi : \mathbb{R}^2 \to \mathbb{S}^2$ be the (inverse of) stereographic projection and take $u_k(x,y) = \pi(kx, ky)$. Since u_k is harmonic for all k we know it solves (see [Hél02])

$$-\Delta u_k = \nabla^\perp B_k.\nabla u_k$$

where $\nabla^\perp (B_k)^i_j = (u_k)^i\nabla(u_k)^j - (u_k)^j\nabla(u_k)^i$, therefore $\|\nabla^\perp B_k\|_{L^2(\mathbb{R}^2)} \leq C\|\nabla u_k\|_{L^2(\mathbb{R}^2)}$. Letting $f_k = \nabla^\perp B_k . \nabla u_k$ we have

$$\|f_k\|_{\mathscr{H}^1(\mathbb{R}^2)} \le C\|\nabla u_k\|^2_{L^2(\mathbb{R}^2)} \le C\|\nabla \pi\|^2_{L^2(\mathbb{R}^2)} = \Lambda < \infty.$$

This tells us two things: first that $\|f_k\|_{h^1(B_1)} \le \Lambda < \infty$ and second that $\|u_k\|_{W^{1,2}(B_1)} \le \Lambda < \infty$. At this point we have all the hypotheses of the theorem (except that we allow $f_k \in h^1$), but it is easy to see that there can be no subsequence converging locally strongly in $W^{1,2}$ because this sequence forms a bubble at the origin.

5.7 Proof of the generalised compact embedding, Theorem 5.2.4

We require a Lemma from functional analysis to help here as we will see that Theorem 5.2.4 is really due to the fact that $W_0^{1,n} \subset\subset Exp^\alpha$ for all $\alpha < \frac{n}{n-1}$ (see Theorem 2.7.9).

Lemma 5.7.1. *Let X, Y be Banach spaces, then $X \subset\subset Y$ implies that $Y^* \subset\subset X^*$.*

Proof. We first point out that for any $\eta > 0$ there exist $N = N(\eta)$ and $\{y_1, ..., y_N\} \subset Y$ such that $B_1^X \subset \bigcup_{i=1}^N B_\eta^Y(y_i)$ which follows from $X \subset\subset Y$.

Second notice that the continuous embedding $Y^* \hookrightarrow X^*$ is trivial.

Now let $y_k^* \subset Y^*$ be a bounded sequence. By Banach-Alaoglu there is a subsequence (not re-labelled) and $y^* \in Y^*$ such that $y_k^* \overset{*}{\rightharpoonup} y^*$.

Pick $\varepsilon > 0$ and let $\eta := \frac{\varepsilon}{\sup_k\{\|y_k^* - y^*\|_{Y^*}\}}$ with $\{y_1, ..., y_N\} \subset Y$ as above, and compute

$$\begin{aligned} \|y_k^* - y^*\|_{X^*} &:= \sup_{x \in X,\, \|x\|_X \le 1} (y_k^* - y^*)(x) \\ &= \sup_{x \in X,\, \|x\|_X \le 1} (y_k^* - y^*)(x - y_{i(x)}) + (y_k^* - y^*)(y_{i(x)}) \end{aligned} \tag{5.2}$$

where $y_{i(x)}$ denotes the closest point in the set $y_1, ...y_N$ to x. In particular we have $\|x - y_{i(x)}\|_Y \le \eta$ thus

$$\|y_k^* - y^*\|_{X^*} \le \varepsilon + \sup_{x \in X,\, \|x\|_X \le 1} (y_k^* - y^*)(y_{i(x)}) \to \varepsilon \tag{5.3}$$

as $k \to \infty$. This holds for all $\varepsilon > 0$ so we are done. □

Recall also the canonical continuous embedding $X \hookrightarrow X^{**}$, thus an immediate Corollary of Lemma 5.7.1 is that $X \hookrightarrow X^{**} \subset\subset Y^{**}$ so that we are left with

Lemma 5.7.2. *Let X and Y be Banach spaces such that Y is reflexive. Then $X \subset\subset Y$ if and only if $Y^* \subset\subset X^*$.*

Proof of Theorem 5.2.4. The space $W_0^{1,n}(U)$ is reflexive, therefore we can see that $W_0^{1,n}(U) = ((W_0^{1,n}(U))^*)^*$. Set $Y = W_0^{1,n}(U)^*$ and $X = L\log^\beta L$. By section 2.7 we know that $X^* = (L\log^\beta L(U))^* = Exp^{\frac{1}{\beta}}(U)$. Therefore by Theorem 2.7.9 we have that,

$$((W_0^{1,n}(U))^*)^* = W_0^{1,n}(U) \subset\subset (L\log^\beta L(U))^*$$

for all $\frac{1}{\beta} < \frac{n}{n-1}$. Therefore by Lemma 5.7.2 we have

$$L\log^\beta L(U) \subset\subset (W_0^{1,n}(U))^*$$

for all $\beta > 1 - \frac{1}{n}$. $\square$

Chapter 6

A generalisation to higher dimensions following Rivière-Struwe

6.1 Introduction

We will consider the following system for maps $u \in M_1^{2,n-2}(B_1)$ (functions with $u, \nabla u$ in the Morrey space $M^{2,n-2}(B_1,\mathbb{R}^m)$), $\omega \in M^{2,n-2}(B_1, so(m) \otimes \wedge^1 \mathrm{T}^*\mathbb{R}^n)$ and $f \in L^p(B_1,\mathbb{R}^m)$ for $\frac{n}{2} < p < n$,

$$-\Delta u = \omega.\nabla u + f, \tag{6.1}$$

where (6.1) is satisfied in a weak sense and $B_1 \subset \mathbb{R}^n$ is the unit ball. For the definition of Morrey spaces see section 2.8. The notation $\omega.\nabla u$ corresponds to both an inner product of one-forms and matrix multiplication, so it reads $(\omega.\nabla u)^i = \langle \omega^i_j, \nabla u^j \rangle$ where we sum over j and $\langle , \rangle$ is the inner product of one-forms induced by the Euclidean metric. Estimating the right hand side of (6.1) using Hölder's inequality leaves us with Δu in the Morrey space $M^{1,n-2}$ (=L^1 when $n = 2$), and the best we can do using singular integral estimates is to conclude that $\nabla u \in M^{(2,\infty),n-2}$ (= $L^{(2,\infty)}$ when $n = 2$). (See section 2.8 for definitions and results if necessary.) These spaces are slightly worse than the spaces we started with, therefore we have lost some information and bootstrapping fails. The anti-symmetry condition on ω is therefore the key to unlocking hidden regularity of this system as first noticed by Rivière [Riv07].

For $n \geq 3$ Rivière-Struwe [RS08] showed that we can find a Coulomb gauge in the Morrey space setting (see Section 4.12), and that this is enough to conclude partial regularity for general solutions. Again this comes down to the appearance of terms that lie in the Hardy space $\mathscr{H}^1$. It is shown that solutions to (6.1) describe harmonic (and almost harmonic) maps from the Euclidean ball into arbitrary Riemannian manifolds. As outlined in [RS08] it would be difficult to carry out the same techniques when $n \geq 3$

as in the case $n = 2$, however Laura Keller [Kel10] has shown that when ω and ∇u lie in a (slightly more restrictive) Besov-Morrey space, then the methods as in the two dimensional case apply.

The regularity obtained in [RS08] and [Rup08] is as follows (see also [Sch10]):

Theorem 6.1.1. *Let u, ω and f be as in* (6.1)*. Then there exists $\epsilon = \epsilon(n, m, p)$ such that whenever $\|\omega\|^2_{M^{2,n-2}(B_1)} \leq \epsilon$ then $u \in C^{0,\gamma}_{loc}$ where $\gamma = 2 - \frac{n}{p} \in (0, 1)$.*

The optimal Hölder regularity was shown in [Rup08] along with an estimate. To see the optimality just consider the case $\omega \equiv 0$; we have that $u \in W^{2,p}_{loc} \hookrightarrow C^{0,2-\frac{n}{p}}_{loc}$ when $\frac{n}{2} < p < n$ by Calderon-Zygmund theory and Morrey estimates.

As stated in [RS08], this theorem allows us to extend the regularity theory for stationary harmonic maps from the Euclidean ball into closed C^2 Riemannian manifolds immersed in some Euclidean space. More precisely it is possible to show that away from a singular set S with $\mathscr{H}^{n-2}(S) = 0$, then any weakly stationary harmonic map is smooth. This follows by a classical theorem stating that continuous weakly harmonic maps are smooth.

6.2 Results

In this chapter we will show improved regularity along with a new estimate when $n \geq 3$. In order to get this estimate we use the Coulomb gauge obtained in [RS08], Theorem 6.1.1 and we crucially require an extension of a result of Adams [Ada75] Proposition 2.8.4. Essentially we would like to estimate $\|\nabla v\|_{L^2}$ for solutions to

$$\Delta v = f \in \mathscr{H}^1$$

however this is not possible unless $n = 2$ ($\mathscr{H}^1 \hookrightarrow H^{-1}$ in this case). However in higher dimensions we know that $f \in \mathscr{H}^1 \cap L^{1,n-2} \hookrightarrow H^{-1}$ for all n (see Remark 2.8.5 part 2).

The key element in this approach is to use the Hölder regularity already obtained in order to get a decay estimate on $\|\nabla u\|_{L^2}$ (Propositions 6.4.1 and 6.3.1) which is an improvement as in all previous work the decay estimates are on $\|\nabla u\|_{L^q}$ for $q < 2$. Once we have this improved decay we show that it is possible to employ a bootstrapping argument using Proposition 2.8.2.

Theorem 6.2.1. *For $n \geq 2$ let $u \in M^{2,n-2}_1(B_1, \mathbb{R}^m)$, $\omega \in M^{2,n-2}(B_1, so(m) \otimes \wedge^1 \mathrm{T}^*\mathbb{R}^n)$ and $f \in L^p(B_1)$ ($p \in (\frac{n}{2}, n)$) weakly solve*

$$-\Delta u = \omega.\nabla u + f.$$

Then there exist $\epsilon = \epsilon(n,m,p)$ *and* $C = C(n,m,p)$ *such that whenever* $\|\omega\|_{M^{2,n-2}(B_1)} \le \epsilon$ *then* $\nabla^2 u \in M^{\frac{2p}{n},n-2}(B_{\frac{1}{2}})$, $\nabla u \in M^{\frac{2p}{n-p},n-2}(B_{\frac{1}{2}})$ *with*

$$\|\nabla^2 u\|_{M^{\frac{2p}{n},n-2}(B_{\frac{1}{2}})} + \|\nabla u\|_{M^{\frac{2p}{n-p},n-2}(B_{\frac{1}{2}})} \le C(\|u\|_{L^1(B_1)} + \|f\|_{L^p(B_1)}).$$

We see that this generalises Theorem 4.2.1 to higher dimensions, moreover it reproves the result in two dimensions without the need to perturb the Coulomb frame. We remark that if $\nabla u \in M^{\frac{2p}{n-p},n-2}$ then $u \in C^{0,\gamma}$ with γ as in Theorem 6.1.1. An interesting question here is whether the integrability of ∇u can be improved further when $n \ge 3$. One might expect that we should have estimates on ∇u in $L^{\frac{np}{n-p}}$ (consider the case $\omega \equiv 0$). Clearly the case $n = 2$ is no problem as this gives the (optimal) regularity expected, moreover we have found solutions with $f \equiv 0$ that are not in $W^{2,2}$ or even $W^{2,(2,\infty)}$. Thus we cannot expect that $\nabla u \in L^\infty$ or even $\nabla u \in BMO$ in general. This also explains the range of f that we consider.

An easy consequence of Theorem 6.2.1 is the following

Corollary 6.2.2. *For* $n \ge 2$ *let* $u \in M_1^{2,n-2}(B_1,\mathbb{R}^m)$, $\omega \in M^{2,n-2}(B_1, so(m)\otimes\wedge^1 \mathrm{T}^*\mathbb{R}^n)$ *weakly solve*

$$-\Delta u = \omega.\nabla u.$$

For any $q < \infty$ *(*$s = \frac{2q}{2+q} < 2$*) there exist* $\epsilon = \epsilon(q,m,n)$ *and* $C = C(q,m,n)$ *such that if* $\|\omega\|_{M^{2,n-2}(B_1)} \le \epsilon$ *then*

$$\|\nabla^2 u\|_{M^{s,n-2}(B_{\frac{1}{2}})} + \|\nabla u\|_{M^{q,n-2}(B_{\frac{1}{2}})} \le C\|u\|_{L^1(B_1)}.$$

Remark 6.2.3. In the case that $|\omega| \le C|\nabla u|$ this automatically gives that when $\|\nabla u\|_{M^{2,n-2}(B_1)}$ is small enough then $u \in W^{2,q}$ for some $q > n$ yielding $u \in C^{1,\gamma}$ for some $\gamma \in (0,1)$. If we knew that ω depended on u and ∇u in a smooth way then we could immediately conclude smoothness by a simple bootstrapping argument using Schauder theory. Thus we recover a proof of the regularity of weakly stationary harmonic maps into Riemannian manifolds (away from a singular set S with $\mathscr{H}^{n-2}(S) = 0$). We also mention that passing to the standard local estimates for smooth harmonic maps also easily follows.

Remark 6.2.4. We remark that Theorem 6.3.1 and Corollary 6.2.2 should hold (with some added technicalities) given any smooth metric g on B_1, with u and ω as above weakly solving

$$-\Delta_g u = \langle \omega, \nabla u\rangle_g.$$

Remark 6.2.5. We also mention here that there are other ways of extending the results of Theorem 6.2.1 and Corollary 6.2.2, for instance one can relax the anti-symmetry condition on the whole of ω and only assume that $\mathrm{d}a$ is anti-symmetric where we have a Hodge decomposition $\omega = \mathrm{d}a + \mathrm{d}^* b + h$.

Another direction would be assume that $u \in W^{1,2}(B_1, \mathbb{C}^m)$ and that $\omega \in L^2(B_1, \mathfrak{g}_{\mathbb{C}} \otimes \wedge^1 \mathrm{T}^* \mathbb{R}^n)$.

Some of these extensions will be explored in a forthcoming work joint with Miaomiao Zhu, where we will use them to improve the known regularity theory for Dirac harmonic maps.

6.3 Proof of the main results, Theorems 6.2.1 and 4.2.1

We prove Theorem 6.2.1 based on the following Proposition, analogous to Lemma 4.6.1, the proof of which is left to section 6.4. This section is the bootstrapping part of the argument, i.e. once we are above a certain regularity we can keep going! In the two dimensional case we use Rivière's gauge in order to gain this extra decay (Lemma 4.6.1), and this is where we will prove Theorem 4.2.1 also.

Proposition 6.3.1. *Let $n \geq 2$ and $u \in M_1^{2,n-2}(B_1, \mathbb{R}^m)$, $\omega \in M^{2,n-2}(B_1, so(m) \otimes \wedge^1 \mathrm{T}^* \mathbb{R}^n)$ and $f \in L^p(B_1, \mathbb{R}^m)$ for $\frac{n}{2} < p < n$, where $B_1 \subset \mathbb{R}^n$ is the unit ball. Now suppose that u is a weak solution to*

$$-\Delta u = \omega . \nabla u + f$$

on B_1. Then there exits $\epsilon = \epsilon(n,m,p)$ such that whenever $\|\omega\|_{M^{2,n-2}(B_1)} < \epsilon$ then $\nabla u \in M_{loc}^{2,n-2(\frac{n}{p}-1)}(B_1, \mathbb{R}^m)$.

Proof of Theorems 6.2.1 and 4.2.1. This proof generalises the ideas needed in the proofs of [ST11, Lemmata 7.1 and 7.2] to Morrey spaces. Using Proposition 6.3.1 and the Hölder inequality we see that $\omega.\nabla u \in M_{loc}^{1,n(1-\frac{1}{p})}$ since (for appropriate B_R)

$$\begin{aligned} \|\omega.\nabla u\|_{L^1(B_R)} &\leq \|\omega\|_{L^2(B_R)} \|\nabla u\|_{L^2(B_R)} \\ &\leq CR^{\frac{n}{2}-1} R^{\frac{n}{2}-\frac{n}{p}+1}. \end{aligned}$$

We can check that the same holds for f (see appendix B) so by Theorem 2.8.2 (weak estimate) we see that this implies $\nabla u \in M_{loc}^{(\frac{n}{n-p},\infty),n(1-\frac{1}{p})}$, which in turn (by a scaling argument) gives, for any $\theta < 1$, $\nabla u \in M_{loc}^{\frac{\theta n}{n-p},n(1-\frac{\theta}{p})}$.

To see this use appendix B in order to consider u on $B_R(x_0)$ by $\hat{u}$ on B_1, we have $\|\nabla \hat{u}\|_{L^{\frac{\theta n}{n-p}}(B_1)} \leq C \|\nabla \hat{u}\|_{L^{\frac{n}{n-p},\infty}(B_1)}$ for any $\theta < 1$, then reversing the scaling gives

$$\|\nabla u\|_{L^{\frac{\theta n}{n-p}}(B_R(x_0))} \leq CR^{\frac{n-p}{\theta}-(n-p)} \|\nabla u\|_{L^{\frac{n}{n-p},\infty}(B_R(x_0))}$$

thus

$$\begin{aligned}
\|\nabla u\|^{\frac{\theta n}{n-p}}_{L^{\frac{\theta n}{n-p}}(B_R(x_0))} &\leq CR^{n(1-\theta)}\|\nabla u\|^{\frac{\theta n}{n-p}}_{L^{\frac{n}{n-p},\infty}(B_R(x_0))}\\
&\leq CR^{n(1-\theta)}R^{\theta n(1-\frac{1}{p})}\|\nabla u\|^{\frac{\theta n}{n-p}}_{M^{(\frac{n}{n-p},\infty),n(1-\frac{1}{p})}}\\
&= CR^{n(1-\frac{\theta}{p})}\|\nabla u\|^{\frac{\theta n}{n-p}}_{M^{(\frac{n}{n-p},\infty),n(1-\frac{1}{p})}}.
\end{aligned}$$

The fact that $\nabla u \in M_{loc}^{\frac{\theta n}{n-p},n(1-\frac{\theta}{p})}$ implies $\omega.\nabla u \in M_{loc}^{s,n(1-\frac{s}{p})}$ where $\frac{1}{s}=\frac{1}{2}+\frac{n-p}{\theta n}$, since (for appropriate B_R and $1=\frac{s}{2}+\frac{s(n-p)}{\theta n}$)

$$\begin{aligned}
\|\omega.\nabla u\|^s_{L^s(B_R)} &\leq \|\omega\|^s_{L^2(B_R)}\|\nabla u\|^s_{L^{\frac{\theta n}{n-p}}(B_R)}\\
&\leq CR^{\frac{ns}{2}-s}R^{ns(1-\frac{\theta}{p})\frac{n-p}{\theta n}}\\
&= CR^{n(1-\frac{s}{p})}.
\end{aligned}$$

We can choose θ so that $s>1$ but note that we also have $s<\frac{2n}{3n-2p}<\frac{2p}{n}$ for $p\in(\frac{n}{2},n)$.

We make the following claim:

$$\omega.\nabla u \in M_{loc}^{s_k,n(1-\frac{s_k}{p})}, s_k\in(1,\frac{2p}{n}) \Rightarrow$$
$$\omega.\nabla u \in M_{loc}^{s_{k+1},n(1-\frac{s_{k+1}}{p})}, s_k<s_{k+1}\in(1,\frac{2p}{n})$$

and $s_{k+1}=\frac{2ns_k}{ns_k+2(n-p)}$.

Before we start we can check that $f\in M^{s_k,n(1-\frac{s_k}{p})}$ with a uniform estimate for any s_k (see appendix B). Therefore we may apply Theorem 2.8.2 (strong estimate) to yield $\nabla u\in M_{loc}^{s_k\frac{n}{n-p},n(1-\frac{s_k}{p})}$, again by Hölder's inequality we have $\omega.\nabla u\in M_{loc}^{s_{k+1},n(1-\frac{s_{k+1}}{p})}$ where $\frac{1}{s_{k+1}}=\frac{1}{2}+\frac{n-p}{s_k n}$ since

$$\begin{aligned}
\|\omega.\nabla u\|^{s_{k+1}}_{L^{s_{k+1}}(B_R)} &\leq \|\omega\|^{s_{k+1}}_{L^2(B_R)}\|\nabla u\|^{s_{k+1}}_{L^{s_k\frac{n}{n-p}}(B_R)}\\
&\leq CR^{\frac{n-2}{2}s_{k+1}}R^{n(1-\frac{s_k}{p})\frac{n-p}{s_k n}s_{k+1}}\\
&= CR^{n(\frac{s_{k+1}}{2}+s_{k+1}\frac{n-p}{s_k n})-s_{k+1}-\frac{n-p}{p}s_{k+1}}\\
&= CR^{n-n\frac{s_{k+1}}{p}}.
\end{aligned}$$

We check that

$$\frac{s_k}{s_{k+1}}=\frac{s_k}{2}+\frac{n-p}{n}<\frac{p}{n}+1-\frac{p}{n}=1.$$

If we assume (to get a contradiction) that $s_{k+1} \geq \frac{2p}{n}$ then we have

$$\frac{2ns_k}{ns_k + 2(n-p)} \geq \frac{2p}{n}$$

which implies

$$2ns_k \geq 2ps_k + 2(n-p)\frac{2p}{n}$$

and therefore $s_k \geq \frac{2p}{n}$, a contradiction. Thus the claim holds.

We have the recursive relation $s_{k+1} = \frac{2ns_k}{ns_k+2(n-p)}$, so we have $s_k \uparrow \frac{2p}{n}$ and we have proved that $\omega.\nabla u \in M_{loc}^{s,n(1-\frac{s}{p})}$ for all $s < \frac{2p}{n}$. Thus we also have $\nabla u \in M_{loc}^{s\frac{n}{n-p},n(1-\frac{s}{p})}$ for all s in this range (see Theorem 2.8.2).

We note here that for $1 < s < t < \frac{2p}{n}$ we have the estimate

$$\|\nabla u\|_{M^{s\frac{n}{n-p},n(1-\frac{s}{p})}} \leq C\|\nabla u\|_{M^{t\frac{n}{n-p},n(1-\frac{t}{p})}}$$

for $C = C(n,p)$ since

$$\begin{aligned}
\|\nabla u\|^{s\frac{n}{n-p}}_{L^{s\frac{n}{n-p}}(B_R)} &\leq CR^{n(\frac{n-p}{sn}-\frac{n-p}{tn})\frac{sn}{n-p}}\|\nabla u\|^{s\frac{n}{n-p}}_{L^{t\frac{n}{n-p}}(B_R)} \\
&\leq C\|\nabla u\|^{s\frac{n}{n-p}}_{M^{t\frac{n}{n-p},n(1-\frac{t}{p})}} R^{n(\frac{n-p}{sn}-\frac{n-p}{tn})\frac{sn}{n-p}} R^{n(1-\frac{t}{p})\frac{s}{t}} \\
&= C\|\nabla u\|^{s\frac{n}{n-p}}_{M^{t\frac{n}{n-p},n(1-\frac{t}{p})}} R^{n(1-\frac{s}{p})}.
\end{aligned}$$

Let $s \in (\frac{2p+n}{2n}, \frac{2p}{n})$ and denote by t the next value given in the bootstrapping argument (if $s = s_k$ then $t = s_{k+1}$). Suppose $\nabla u \in M^{s\frac{n}{n-p},n(1-\frac{s}{p})}$ giving $\omega.\nabla u \in M^{s,n(1-\frac{s}{p})}$. Notice that within this range, by Remark 2.8.5 there is a $C = C(n,p)$ independent of s and t such that

$$\begin{aligned}
\|\nabla u\|_{M^{t\frac{n}{n-p},n(1-\frac{t}{p})}(B_{\frac{1}{2}})} &\leq C(\|\omega\|_{M^{2,n-2}(B_1)}\|\nabla u\|_{M^{s\frac{n}{n-p},n(1-\frac{s}{p})}(B_1)} + \|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)}) \\
&\leq C(\|\omega\|_{M^{2,n-2}(B_1)}\|\nabla u\|_{M^{t\frac{n}{n-p},n(1-\frac{t}{p})}(B_1)} + \|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)}).
\end{aligned}$$

Raising to the power $\mu := t\frac{n}{n-p}$ we see that

$$\|\nabla u\|^{\mu}_{M^{\mu,n(1-\frac{t}{p})}(B_{\frac{1}{2}})} \leq C(\|\omega\|^{\mu}_{M^{2,n-2}(B_1)}\|\nabla u\|^{\mu}_{M^{\mu,n(1-\frac{t}{p})}(B_1)} + (\|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)})^{\mu}).$$

where we can still pick C independent of t since $\mu < \frac{2p}{n-p}$.

Now, rescaling about some ball $B_R(x_0) \subset B_1$ gives us (see appendix B)

$$\|\nabla \hat{u}\|^{\mu}_{M^{\mu,n(1-\frac{t}{p})}(B_{\frac{1}{2}})} \leq C(\|\hat{\omega}\|^{\mu}_{M^{2,n-2}(B_1)}\|\nabla \hat{u}\|^{\mu}_{M^{\mu,n(1-\frac{t}{p})}(B_1)} + (\|\hat{f}\|_{L^p(B_1)} + \|\hat{u}\|_{L^1(B_1)})^{\mu}).$$

Undoing the scaling leaves

$$\begin{aligned} R^{\mu+\frac{nt}{p}}\|\nabla u\|^{\mu}_{M^{\mu,n(1-\frac{t}{p})}(B_{\frac{R}{2}}(x_0))} &\leq C(\|\omega\|^{\mu}_{M^{2,n-2}(B_1)}R^{\mu+\frac{nt}{p}}\|\nabla u\|^{\mu}_{M^{\mu,n(1-\frac{t}{p})}(B_R(x_0))} + \\ &+ (R^{2-\frac{n}{p}}\|f\|_{L^p(B_1)} + R^{-n}\|u\|_{L^1(B_1)})^{\mu}). \end{aligned}$$

Since $R<1$, $\mu < \frac{2p}{n-p}$ and $t < \frac{2p}{n}$ we have that

$$\begin{aligned} \|\nabla u\|^{\mu}_{M^{\mu,n(1-\frac{t}{p})}(B_{\frac{R}{2}}(x_0))} &\leq C\|\omega\|^{\mu}_{M^{2,n-2}(B_1)}\|\nabla u\|^{\mu}_{M^{\mu,n(1-\frac{t}{p})}(B_R(x_0))} + \\ &+ C(\|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)})^{\mu})R^{-((n+1)\frac{2p}{n-p}+2)}. \end{aligned}$$

We are now in a position to apply Lemma C.0.5 for $\Gamma = C(\|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)})^{\mu})$, $\epsilon \leq \left(\frac{\epsilon_0}{C}\right)^{\frac{n-p}{2p}}$ and $\epsilon_0 = \epsilon_0(n,k)$ is found for $k = (n+1)\frac{2p}{n-p}+2$ to give the estimate

$$\|\nabla u\|_{M^{\mu,n(1-\frac{t}{p})}(B_{\frac{1}{2}})} \leq C(\|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)})$$

with C independent of t. We may now pass to the limit $t \uparrow \frac{2p}{n}$ to give

$$\|\nabla u\|_{M^{\frac{2p}{n-p},n-2}(B_{\frac{1}{2}})} \leq C(\|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)}).$$

For the second estimate, note that we have $\omega.\nabla u \in M^{\frac{2p}{n},n-2}$ by Hölder's inequality, thus by Theorem 2.8.1 and the proceeding remarks we have finished the proof.

□

6.4 Proof of the improved decay, Proposition 6.3.1

We begin with a proposition stating the main decay estimate required, the proof of this is left until section 6.5. This decay estimate is analogous to that of part 2. from Theorem 4.2.2, except that here we crucially require the Hölder regularity already obtained in order to prove (6.2).

Proposition 6.4.1. *With the set-up as in Proposition 6.3.1. Let $\delta > 0$, then there exist*

$\epsilon = \epsilon(n,m,p) > 0$ small enough and $C = C(\delta,m,n)$ such that when $\|\omega\|_{M^{2,n-2}(B_1)} \le \epsilon$ we have the following estimate ($\gamma = 2-\frac{n}{p}$)

$$\|\nabla u\|^2_{L^2(B_r)} \le C(\delta)(\|\omega\|^2_{M^{2,n-2}(B_1)}[u]^2_{C^{0,\gamma}(B_1)} + \|f\|^2_{L^p(B_1)}) + r^n(1+\delta)\|\nabla u\|^2_{L^2(B_1)}. \tag{6.2}$$

Proof of Proposition 6.3.1. We follow the argument for the proof of [ST11, Lemma 7.3]: Pick $\delta = \delta(n,p)$ sufficiently small so that

$$\lambda := \frac{1+\delta}{2^n} < \frac{1}{2^{n-2+2\gamma}} := \Lambda \in \left(\frac{1}{2^n}, 1\right) \tag{6.3}$$

since $\gamma = 2-\frac{n}{p} \in (0,1)$.

Consider the solution on some small ball $B_R(x_0) \subset B_1$. We have (by (6.2) and appendix B)

$$\|\nabla \hat{u}\|^2_{L^2(B_r)} \le C(\|\omega\|^2_{M^{2,n-2}(B_1)}[\hat{u}]^2_{C^{0,\gamma}(B_1)} + \|\hat{f}\|^2_{L^p(B_1)}) + r^n(1+\delta)\|\nabla \hat{u}\|^2_{L^2(B_1)},$$

and setting $r = \frac{1}{2}$ yields

$$\|\nabla \hat{u}\|^2_{L^2(B_{\frac{1}{2}})} \le \lambda\|\nabla \hat{u}\|^2_{L^2(B_1)} + C(\|\omega\|^2_{M^{2,n-2}(B_1)}[\hat{u}]^2_{C^{0,\gamma}(B_1)} + \|\hat{f}\|^2_{L^p(B_1)}).$$

Undoing the scaling gives

$$\|\nabla u\|^2_{L^2(B_{\frac{R}{2}}(x_0))} \le \lambda\|\nabla u\|^2_{L^2(B_R)} + CR^{n-2+2\gamma}(\|\omega\|^2_{M^{2,n-2}(B_1)}[u]^2_{C^{0,\gamma}(B_R(x_0))} + \|f\|^2_{L^p(B_R(x_0))}).$$

Therefore, setting $R = 2^{-k}$, $k \in \mathbb{N}_0$ and $a_k := \|\nabla u\|^2_{L^2(B_{2^{-k}})}$ we have

$$\begin{aligned} a_{k+1} &\le \lambda a_k + \left(\frac{1}{2}\right)^{k(n-2+2\gamma)} C(\|\omega\|^2_{M^{2,n-2}(B_1)}[u]^2_{C^{0,\gamma}(B_1)} + \|f\|^2_{L^p(B_1)}) \\ &= \lambda a_k + \Lambda^k C(\|\omega\|^2_{M^{2,n-2}(B_1)}[u]^2_{C^{0,\gamma}(B_1)} + \|f\|^2_{L^p(B_1)}). \end{aligned}$$

This can be solved to yield (letting $K := C(\|\omega\|^2_{M^{2,n-2}(B_1)}[u]^2_{C^{0,\gamma}(B_1)} + \|f\|^2_{L^p(B_1)})$)

$$a_{k+1} \le \lambda^k a_1 + K\Lambda\frac{(\Lambda^k - \lambda^k)}{\Lambda - \lambda},$$

and by (6.3), this simplifies to

$$\|\nabla u\|^2_{L^2(B_{2^{-k}}(x_0))} =: a_k \le C\Lambda^k$$

Thus, for $r \in (0,1/2]$ we have

$$\|\nabla u\|^2_{L^2(B_r(x_0))} \le C r^{n-2+2\gamma}.$$

We have

$$\|\nabla u\|^2_{M^{2,n-2+2\gamma}(B_{\frac{1}{2}})} \le C.$$

and a covering argument concludes the proof.

□

6.5 Proof of the decay estimate, Proposition 6.4.1

Proof of Proposition 6.4.1. We will use the Coulomb gauge in order to re-write our equation, so set ϵ small enough so that we can apply Theorem 3.2.3. We have (see appendix A for the relevant background on Sobolev forms)

$$\mathrm{d}^*(P^{-1}\mathrm{d}u) = \langle \mathrm{d}^*\eta, P^{-1}\mathrm{d}u\rangle + P^{-1}f$$

and

$$\mathrm{d}(P^{-1}\mathrm{d}u) = (\mathrm{d}P^{-1} \wedge \mathrm{d}u).$$

We can also set ϵ small enough in order to apply Theorem 6.1.1 so that $u \in C^{0,\gamma}$ where $\gamma = 2 - \frac{n}{p}$. Now we wish to extend the quantities arising above in the appropriate way: First of all we may extend η by zero. We also extend $P - \fint_{B_1} P$ to $\tilde{P} \in W^{1,2} \cap L^\infty(\mathbb{R}^n)$ and finally u to $\tilde{u} \in C^{0,\gamma}(\mathbb{R}^n)$ where each has compact support in B_2 (we may assume $u \in C^{0,\gamma}(\overline{B}_1)$).

Note that we have $\|\nabla \tilde{P}\|_{L^2} \le C\|\nabla P\|_{L^2(B_1)} \le C\|\omega\|_{M^{2,n-2}(B_1)}$ by Poincaré's inequality and $\nabla\tilde{P} = \nabla P$ in B_1. We also have $\tilde{u} \in C^{0,\gamma}(\mathbb{R}^n)$ with $\|\tilde{u}\|_{C^{0,\gamma}} \le C\|u\|_{C^{0,\gamma}}$ and (since we may assume $\int u = 0$) we have $\|\tilde{u}\|_{C^{0,\gamma}} \le C[u]_{C^{0,\gamma}}$, moreover $\tilde{u} = u$ in B_1. All the constants here come from standard extension operators and are independent of the function, see for instance [GT01].

Now we use Lemma A.0.2 in order to write $P^{-1}\mathrm{d}u = \mathrm{d}a + \mathrm{d}^*b + h$ with a, b, h as in the Lemma. Notice that we have $\Delta a = \langle \mathrm{d}^*\eta, P^{-1}\mathrm{d}u\rangle + P^{-1}f$ and $\Delta b = \mathrm{d}P^{-1} \wedge \mathrm{d}u$ weakly. We proceed to estimate $\nabla u \in L^2$ by estimating $\|\mathrm{d}a\|_{L^2}$, $\|\mathrm{d}^*b\|_{L^2}$ and using standard properties of harmonic functions in order to deal with $\|h\|_{L^2}$.

We start with $\|\mathrm{d}a\|_{L^2}$; notice that $\langle \mathrm{d}^*\eta, P^{-1}\mathrm{d}u\rangle = \langle \mathrm{d}^*\eta, \mathrm{d}(P^{-1}u)\rangle - \langle \mathrm{d}^*\eta, \mathrm{d}P^{-1}\rangle u =$

$I+II$. For I, pick $\phi \in C_c^\infty(B_1)$ and check (we use that η has zero boundary values)

$$\begin{aligned}\int * \langle d^*\eta, d(P^{-1}u)\rangle \phi &= (d^*\eta, d(P^{-1}u)\phi)\\ &= (d^*\eta, d(P^{-1}u\phi)) - (d^*\eta, (d\phi)P^{-1}u)\\ &= -(d^*\eta, (d\phi)P^{-1}u)\\ &\leq \|\nabla\eta\|_{L^2(B_1)}\|\nabla\phi\|_{L^2(B_1)}\|P^{-1}u\|_{L^\infty(B_1)}\\ &\leq C\|\omega\|_{M^{2,n-2}(B_1)}\|\nabla\phi\|_{L^2(B_1)}[u]_{C^{0,\gamma}(B_1)}.\end{aligned}$$

We have $I \in H^{-1}(B_1)$ with

$$\|I\|_{H^{-1}(B_1)} \leq C\|\omega\|_{M^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)}. \tag{6.4}$$

For II notice that $\langle d^*\eta, d\tilde{P}^{-1}\rangle \tilde{u} = \langle d^*\eta, dP^{-1}\rangle u$ in B_1. Moreover $\langle d^*\eta, d\tilde{P}^{-1}\rangle \in \mathscr{H}^1(\mathbb{R}^n)$ with

$$\|\langle d^*\eta, d\tilde{P}^{-1}\rangle\|_{\mathscr{H}^1} \leq C\|\nabla\eta\|_{L^2(B_1)}\|\nabla\tilde{P}\|_{L^2(B_1)} \leq C\|\omega\|^2_{M^{2,n-2}(B_1)}$$

by the results in [CLMS93] (see section 2.6). Therefore (see Section 2.6) we have

$$\langle d^*\eta, d\tilde{P}^{-1}\rangle \tilde{u} \in h^1(\mathbb{R}^n)$$

with

$$\|\langle d^*\eta, d\tilde{P}^{-1}\rangle \tilde{u}\|_{h^1(\mathbb{R}^n)} \leq C\|\omega\|^2_{M^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)}.$$

We also have $\|M_{n-2}(\langle d^*\eta, d\tilde{P}^{-1}\rangle \tilde{u})\|_{L^\infty} \leq C\|\omega\|^2_{M^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)}$ since for $R>0$

$$R^{2-n}\int_{B_R(x_0)}\langle d^*\eta, d\tilde{P}^{-1}\rangle \tilde{u} = R^{2-n}\int_{B_R(x_0)\cap B_1}\langle d^*\eta, d\tilde{P}^{-1}\rangle \tilde{u} \leq C\|\omega\|^2_{M^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)}$$

(remember η was extended by zero). Now, using the Remark 2.8.5 we have $\langle d^*\eta, d\tilde{P}^{-1}\rangle \tilde{u} \in H^{-1}(B_1)$, moreover $\langle d^*\eta, d\tilde{P}^{-1}\rangle \tilde{u} = \langle d^*\eta, dP^{-1}\rangle u$ in B_1 so

$$\|II\|_{H^{-1}(B_1)} \leq C\|\omega\|^2_{M^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)}. \tag{6.5}$$

Putting (6.4) and (6.5) together yields $\langle d^*\eta, P^{-1}du\rangle \in H^{-1}(B_1)$ with (assuming $\epsilon < 1$)

$$\|\langle d^*\eta, P^{-1}du\rangle\|_{H^{-1}(B_1)} \leq C\|\omega\|_{M^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)}.$$

It is easy to check that $P^{-1}f \in H^{-1}(B_1)$ with $\|P^{-1}f\|_{H^{-1}(B_1)} \leq C\|f\|_{L^p(B_1)}$, overall this means that $a \in W_0^{1,2}(B_1)$ weakly solves

$$\Delta a = \langle \mathrm{d}^*\eta, P^{-1}\mathrm{d}u\rangle + P^{-1}f,$$

so we have

$$\|\nabla a\|_{L^2(B_1)} \leq C(\|\omega\|_{M^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)} + \|f\|_{L^p(B_1)}). \tag{6.6}$$

Now we need to estimate $\|\mathrm{d}^*b\|_{L^2(B_1)}$. We know that $b \in W_N^{1,2}(B_1, \wedge^2\mathbb{R}^n)$ (see appendix A for a definition) has $\mathrm{d}b = 0$ and $\Delta b = (\mathrm{d}P^{-1} \wedge \mathrm{d}u)$. We have

$$\|\mathrm{d}^*b\|_{L^2(B_1)} = \sup_{E\in C^\infty(B_1,\wedge^1\mathbb{R}^n)\ \|E\|_{L^2(B_1)}\leq 1} (\mathrm{d}^*b, E).$$

Using a smooth version of Lemma A.0.2 we can decompose each E by $E = \mathrm{d}e_1 + \mathrm{d}^*e_2 + e_3$ where $e_1 \in C_0^\infty(B_1)$, $e_2 \in C_N^\infty(B_1, \wedge^2\mathbb{R}^n)$ with $\mathrm{d}e_2 = 0$ and $\mathrm{d}e_3 = \mathrm{d}^*e_3 = 0$ (e_3 is a harmonic one form). Notice that $(\mathrm{d}^*b, \mathrm{d}e_1) = 0$ since b has zero normal component and $\mathrm{d}^2e_1 = 0$. Also we have $(\mathrm{d}^*b, e_3) = 0$ since e_3 is harmonic and b has vanishing normal components. Therefore

$$\begin{aligned}
(\mathrm{d}^*b, E) &= (\mathrm{d}^*b, \mathrm{d}^*e_2)\\
&= (P^{-1}\mathrm{d}u, \mathrm{d}^*e_2)\\
&= (\mathrm{d}(P^{-1}u), \mathrm{d}^*e_2) - ((\mathrm{d}P^{-1})u, \mathrm{d}^*e_2)\\
&= -((\mathrm{d}P^{-1})u, \mathrm{d}^*e_2)\\
&\leq C\|\omega\|_{M^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)}\|\mathrm{d}^*e_2\|_{L^2(B_1)}\\
&\leq C\|\omega\|_{M^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)}\|E\|_{L^2(B_1)}.
\end{aligned}$$

Therefore

$$\|\mathrm{d}^*b\|_{L^2(B_1)} \leq C\|\omega\|_{M^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)}. \tag{6.7}$$

We note here that by [Mor66, Theorem 7.5.1] and $\mathrm{d}b = 0$ that we in fact have the same estimate for ∇b.

We now use the fact that h is harmonic giving that the quantity $r^{-n}\|h\|^2_{L^2(B_r)}$ is increasing, and Lemma A.0.2 to give

$$\begin{aligned}
\|h\|^2_{L^2(B_r)} &\leq r^n\|h\|^2_{L^2(B_1)}\\
&\leq r^n\|P^{-1}\mathrm{d}u\|^2_{L^2(B_1)}\\
&= r^n\|\mathrm{d}u\|^2_{L^2(B_1)}
\end{aligned}$$

where the last line follows because P is orthogonal.

Going back to our original Hodge decomposition we see that (using Young's inequality, the orthogonality of P, (6.6) and (6.7))

$$\begin{aligned}
\|\mathrm{d}u\|^2_{L^2(B_r)} &= \|P^{-1}\mathrm{d}u\|^2_{L^2(B_r)} \\
&\leq (\|h\|_{L^2(B_r)} + \|\mathrm{d}a\|_{L^2(B_r)} + \|\mathrm{d}^* b\|_{L^2(B_r)})^2 \\
&\leq (1+\delta)\|h\|^2_{L^2(B_r)} + C_\delta(\|\mathrm{d}a\|_{L^2(B_r)} + \|\mathrm{d}^* b\|_{L^2(B_r)})^2 \\
&\leq (1+\delta) r^n \|\mathrm{d}u\|^2_{L^2(B_1)} + C_\delta(\|\omega\|^2_{M^{2,n-2}(B_1)}[u]^2_{C^{0,\gamma}(B_1)} + \|f\|^2_{L^p(B_1)}).
\end{aligned}$$

This completes the proof.

□

Chapter 7

Critical $\overline{\partial}$ problems for surfaces

7.1 Introduction

Here we will study a related first order system, a $\overline{\partial}$ problem, on a vector bundle over a surface (equipped with a connection). We show optimal conditions on the connection forms which allow one to impose a holomorphic structure. We have already seen a version of this (Theorem 3.3.1), however we will show that one can weaken the hypothesis in order to obtain such a frame. In particular this allows one to prove the full regularity for harmonic maps in the case that the target $\mathcal{N} \hookrightarrow \mathbb{R}^m$ is isometrically embedded and C^3 or $\mathcal{N}$ is C^2 with a trivial normal bundle.

7.2 Results

We will consider a connection on a vector bundle E over a Riemann surface Σ, equipped with a bundle metric. Our vector bundle may be real or complex, and the connection is compatible with the fibre metric. In this setting we choose our trivialisations such that the metric is trivial. Thus our connection forms are skew-hermitian or skew-symmetric one forms. Since we are working over a Riemann surface we may consider the related $\overline{\partial}$-problem associated to sections of $\wedge^{(1,0)}\Sigma \otimes E$. We ask under what circumstances we can impose a holomorphic structure on our vector bundle. I.e. can we always find a local frame such that the transition charts are holomorphic. We note that we will have to complexify E when it is real, thus we will search for a frame in the complex general linear group. If we could find a real frame with these holomorphic transition properties then the connection must be flat! We work with a small piece of Σ over which E is trivial, therefore we may simply consider the case $\Sigma = D \subset \mathbb{C}$ the unit disc.

We consider local connections $\omega \in L^2(D, \mathfrak{g}_{\mathbb{F}} \otimes \wedge^1 \mathrm{T}^* D)$ admitting the following Hodge decomposition:

$$\omega = \mathrm{d}a + \mathrm{d}^* b$$

with $a \in W^{1,2}$, $b \in W^{1,(2,1)}$ and

$$\|\nabla b\|_{L^{2,1}(D)} \leq K.$$

We say that such an ω satisfies condition †. Another way of writing this condition is that $\omega \in L^2(D, \mathfrak{g}_{\mathbb{F}} \otimes \wedge^1 \mathrm{T}^* D)$ satisfies

$$\nabla\Delta^{-1}(\mathrm{d}\omega) \in L^{2,1}$$

with

$$\|\nabla\Delta^{-1}(\mathrm{d}\omega)\|_{L^{2,1}(D)} \leq K.$$

So far we have not considered the domain as complex – effectively we have

$$\omega = \omega^x \mathrm{d}x + \omega^y \mathrm{d}y,$$

with $\omega^x, \omega^y \in L^2(D, \mathfrak{g}_{\mathbb{F}})$ however we can just as well express ω in terms of $\mathrm{d}z$ and $\mathrm{d}\overline{z}$;

$$\omega = \omega^{\overline{z}} + \omega^z$$

where $\omega^{\overline{z}} = \frac{1}{2}(\omega^x + i\omega^y)\mathrm{d}\overline{z} \in L^2(D, gl(m,\mathbb{C}) \otimes \wedge^{(0,1)}\mathrm{T}^*_{\mathbb{C}} D)$ and $\omega^z = \frac{1}{2}(\omega^x - i\omega^y)\mathrm{d}z$.

Theorem 7.2.1. *There exists $\varepsilon > 0$ such that whenever ω satisfies condition † and*

$$\|\omega\|_{L^2} + K \leq \varepsilon$$

there exists a change of frame $S \in L^\infty \cap W^{1,2}(D, Gl(k,\mathbb{C}))$ such that

$$\overline{\partial} S = -\omega^{\overline{z}} S$$

with

$$\|\mathrm{dist(S, U(m))}\|_{\mathrm{L^\infty(D)}} \leq \frac{1}{3}$$

and

$$\|\nabla S\|_{L^2(D_{\frac{1}{2}})} \leq C\|\omega\|_{L^2}.$$

Corollary 7.2.2. *Let $\alpha \in L^2(D, \mathbb{C}^m \otimes \wedge^{(1,0)}\mathrm{T}^*_{\mathbb{C}} D)$ and $\omega \in L^2(D, \mathfrak{g}_{\mathbb{F}} \otimes \wedge^1 \mathrm{T}^* D)$. Suppose that ω satisfies condition † and that $\overline{\partial}_\omega \alpha = 0$, i.e.*

$$\overline{\partial}\alpha = -\omega^{\overline{z}} \wedge \alpha$$

then $\alpha \in L^\infty \cap W^{1,2}(U)$ *for all* $U \subset\subset D$*. When* $\|\omega\|_{L^2} + K$ *is small enough we have*

$$\|\alpha\|_{L^\infty(D_{\frac{1}{3}})} \leq C\|\alpha\|_{L^1}$$

and

$$\|\nabla\alpha\|_{L^2(D_{\frac{1}{3}})} \leq C\|\alpha\|_{L^1}(1 + \|\omega\|_{L^2}).$$

Moreover, under these assumptions we have $|\alpha|^2 \in h^1(D)$ *with*

$$\||\alpha|^2\|_{h^1(D)} \leq C\|\alpha\|^2_{L^2(D)}.$$

Remark 7.2.3. We remark here that given any $\bar{\omega} \in L^2(D, gl(m,\mathbb{C}) \otimes \wedge^{(0,1)}\mathrm{T}^*_{\mathbb{C}}D)$ we can always find a unique $\omega \in L^2(D, \mathfrak{g}_{\mathbb{F}} \otimes \wedge^1\mathrm{T}^*\mathbb{R}^2)$ such that $\omega^{\overline{z}} = \bar{\omega}$. Indeed if we write

$$\bar{\omega} = (\bar{\omega}_1 + i\bar{\omega}_2)\mathrm{d}\overline{z}$$

with $\bar{\omega}_j : D \to gl(m,\mathbb{R})$. Then we can decompose each $\bar{\omega}_j$ into its symmetric and anti-symmetric part,

$$\bar{\omega}_j = \bar{\omega}^S_j + \bar{\omega}^A_j$$

thus letting $\omega^x = 2(\bar{\omega}^A_1 + i\bar{\omega}^S_2) : D \to u(m)$ and $\omega^y = 2(\bar{\omega}^A_2 - i\bar{\omega}^S_1) : D \to u(m)$ and

$$\omega = \omega^x\mathrm{d}x + \omega^y\mathrm{d}y \in L^2(D, \mathfrak{g}_{\mathbb{F}} \otimes \wedge^1\mathrm{T}^*\mathbb{R}^2)$$

we have

$$\omega^{\overline{z}} = \frac{1}{2}(\omega^x + i\omega^y)\mathrm{d}\overline{z} = \bar{\omega}.$$

Therefore for any such $\bar{\omega}$ we can apply Theorem 7.2.1 if ω has a Hodge decomposition of the form described above.

An application of Theorem 7.2.1 and Corollary 7.2.2 is for Harmonic maps from a Riemann surface into a C^3 closed target isometrically embedded in $\mathbb{R}^m$. In this case we have (where $\alpha = \partial u$)

$$\overline{\partial}(\partial u) = -\omega^{\overline{z}} \wedge \partial u$$

where $\omega^i_j = (\mathscr{A}^i_{jk}(u) - \mathscr{A}^j_{ik}(u))\mathrm{d}u^k$ (which gives $(\omega^{\overline{z}})^i_j = (\mathscr{A}^i_{jk}(u) - \mathscr{A}^j_{ik}(u))\overline{\partial}u^k$). We remark here that $\mathrm{d}u$ solves the coupled system

$$\mathrm{d}^*_\omega(\mathrm{d}u) = \mathrm{d}^*\mathrm{d}u - *(\omega \wedge *\mathrm{d}u) = 0$$

and

$$\mathrm{d}_\omega(\mathrm{d}u) = \mathrm{d}(\mathrm{d}u) + \omega \wedge \mathrm{d}u = 0$$

which can be checked directly (compare with section 3.4).

When $\mathcal{N}$ is C^3 we have $\mathscr{A}^i_{jk}(u) \in W^{1,2}$ with

$$\|\nabla \mathscr{A}^i_{jk}(u)\|_{L^2} \leq C(\mathcal{N})\|\nabla u\|_{L^2}.$$

Thus for a Hodge decomposition $\omega = \mathrm{d}a + \mathrm{d}^* b$ with $b \in W^{1,2}_0(D, \wedge^2 \mathrm{T}^* D)$ we have

$$\Delta b^i_j = \mathrm{d}(\mathscr{A}^i_{jk}(u) - \mathscr{A}^j_{ik}(u)) \wedge \mathrm{d}u^k \in \mathscr{H}^1$$

and therefore $\nabla b \in L^{2,1}$ with

$$\|\nabla b\|_{L^{2,1}} \leq C\|\nabla u\|^2_{L^2}$$

and ω satisfies condition † with

$$K + \|\omega\|_{L^2} \leq C\|\nabla u\|_{L^2}$$

whenever $\|\nabla u\|_{L^2} \leq 1$.

Thus, Corollary 7.2.2 immediately gives Lipschitz estimates on u. The full regularity for harmonic maps follows from an easy boot-strapping argument.

The same argument applies even when $\mathcal{N}$ is C^2 under the added assumption that the normal bundle is trivial (for instance when $\mathcal{N}$ is diffeomorphic to a sphere). In this case we can have a global normal frame $\{\nu_k\}^m_{k=m-N}$ for $\mathrm{N}\mathcal{N}$ that is C^1, and we can write

$$\omega^i_j = \sum_k \mathrm{d}\nu^i_k(u)\nu^j_k(u) - \mathrm{d}\nu^j_k(u)\nu^i_k(u)$$

(so that $(\omega^{\bar z})^i_j = \sum_k \bar\partial \nu^i_k(u)\nu^j_k(u) - \bar\partial\nu^j_k(u)\nu^i_k(u)$). Again for a Hodge decomposition as above we have

$$\Delta b^i_j = \sum_k 2\mathrm{d}\nu^i_k(u) \wedge \mathrm{d}\nu^j_k(u) \in \mathscr{H}^1$$

and therefore $\nabla b \in L^{2,1}$ with

$$\|\nabla b\|_{L^{2,1}} \leq C\|\nabla u\|^2_{L^2}.$$

7.3 Counter example in the complex case

Here we present a counter example to show that the condition on the Hodge decomposition is sharp.

Consider $\alpha : D \to \mathbb{C}^2$ given by $\alpha(z) = \frac{1}{z\log(\frac{e}{|z|})}(1,-i)\mathrm{d}z \in L^2(D, \mathbb{C}^2 \otimes \wedge^{(1,0)}\mathrm{T}^*_{\mathbb{C}}D)$ and we define $\omega \in L^2(D, so(2) \otimes \wedge^1 \mathrm{T}^*\mathbb{R}^2)$ by

$$\omega = \frac{1}{r^2\log(\frac{e}{r})}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(y\mathrm{d}x - x\mathrm{d}y)$$

so that

$$\omega^{\overline{z}} = \frac{i/2}{\bar{z}\log(\frac{e}{|z|})}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathrm{d}\overline{z}$$

A short claculation yields that

$$\frac{\partial \alpha}{\partial \bar{z}}\mathrm{d}\overline{z} \wedge \mathrm{d}z = \frac{1/2}{(|z|\log(\frac{e}{|z|}))^2}(1,-i)\mathrm{d}\overline{z} \wedge \mathrm{d}z = -\omega^{\overline{z}} \wedge \alpha.$$

Now, given any Hodge decomposition of $\omega = \mathrm{d}a + \mathrm{d}^* b$ we must have

$$\Delta b = \left(\frac{1}{r\log\left(\frac{e}{r}\right)}\right)^2$$

therefore we cannot possibly have $\nabla b \in L^{2,1}$ since we know that there is a solution to this that is not continuous at the origin (the Frehse example at the end of section 2.3). In fact we have $\mathrm{d}^* b \in L^{2,q}$ for all $q > 1$, showing that the hypotheses in condition † are sharp.

We see that α is a weak solution to the above equation but

$$\frac{\partial \alpha}{\partial z} = \left(\frac{-1}{z^2\log(\frac{e}{|z|})} + \frac{1/2}{(z\log(\frac{e}{|z|}))^2}\right)(1,-i) \notin L^1(D).$$

Thus Theorem 7.2.1 fails in this case: It is easy to see that $\omega = \mathrm{d}^* u$ where $u = \log\log(\frac{e}{r})$ is the Frehse example, thus we can never have a Hodge decomposition of the form required.

7.4 Proof of the regularity result, Corollary 7.2.2

We may always restrict to the case that $\|\omega\|_{L^2} + K \leq \varepsilon$ (given by Theorem 7.2.1). We check that

$$\overline{\partial}(S^{-1}\alpha) = -S^{-1}\overline{\partial}SS^{-1} \wedge \alpha - S^{-1}\omega^{\overline{z}} \wedge \alpha = 0.$$

Therefore $\alpha = Sh$ for some holomorphic h and the estimates follow by standard theory.

The proof of the final assertion (that $|\alpha|^2 \in h^1$) follows from the following fact that is easily verified (see [CM08, Proof of Proposition C.1]): Given a holomorphic function $h \in L^2(D)$, then $|h|^2 \in h^1(D)$ with

$$\||h|^2\|_{h^1(D)} \leq C\|h\|^2_{L^2(D)}.$$

To see this first notice that $h = f_z$ for some holomorphic $f = f_1 + if_2 \in W^{1,2}(D)$ (this follows from the Poincaré lemma, for instance). Thus we have (since f is holomorphic)

$$|h|^2 = |f_z|^2 = -\nabla f_1.\nabla^\perp f_2 \in h^1(D).$$

In our case we have $\alpha = Sh$ so that there exists some C with

$$C^{-1}|h|^2 \leq |\alpha|^2 \leq C|h|^2.$$

Thus we have

$$\||\alpha|^2\|_{h^1(B_1)} \leq C\||h|^2\|_{h^1(B_1)} \leq C\|h\|^2_{L^2(D)} \leq C\|\alpha\|^2_{L^2(D)}$$

where the first inequality can be seen by checking that

$$\tilde{\tilde{M}}_\phi[|\alpha|^2] \leq C\tilde{\tilde{M}}_\phi[|h|^2]$$

when $\phi \geq 0$.

7.5 Proof of the existence of a holomorphic gauge, Theorem 7.2.1

We start by finding the Coulomb frame associated to da; using Theorem 3.2.3 we can find $P \in W^{1,2}(D, \mathfrak{G}_{\mathbb{F}})$ and $\eta \in W^{1,2}_0(D, \mathfrak{g}_{\mathbb{F}} \otimes \wedge^2 \mathrm{T}^* D)$ such that

$$P^{-1}\mathrm{d}P + P^{-1}\mathrm{d}aP = \mathrm{d}^*\eta \tag{7.1}$$

and

$$\|\nabla P\|_{L^2(D)} + \|\eta\|_{W^{1,2}(D)} \leq C\|\mathrm{d}a\|_{L^2(D)} \leq C\|\omega\|_{L^2(D)}.$$

Thus on D we have a solution to

$$\Delta\eta = \mathrm{d}\bar{P}^T \wedge \mathrm{d}P + \mathrm{d}(\bar{P}^T \mathrm{d}aP).$$

Or, in coordinates we have

$$\Delta\eta^i_j = \sum_k \mathrm{d}\bar{P}^k_i \wedge \mathrm{d}P^k_j + \sum_{k,l} \mathrm{d}(\bar{P}^l_i P^k_j) \wedge \mathrm{d}a^l_k \in \mathscr{H}^1.$$

Standard Wente estimates (see for instance Lemma A.1 of [Riv07]) give

$$\|\eta\|_{W^{2,1}(D)} \leq C\|\omega\|_{L^2(D)}$$

which we can couple with Theorem 2.7.2 to give

$$\|\nabla\eta\|_{L^{2,1}(D)} \leq C\|\omega\|_{L^2(D)}.$$

Now we check how P transforms ω, by (7.1) we have

$$P^{-1}\mathrm{d}P + P^{-1}\omega P = \omega_P = \mathrm{d}^*\eta + P^{-1}\mathrm{d}^* bP \in L^{(2,1)}(D) \tag{7.2}$$

and

$$\|\mathrm{d}^*\eta + P^{-1}\mathrm{d}^* bP\|_{L^{2,1}(D)} \leq C(\|\omega\|^2_{L^2(D)} + K).$$

We can see here the significance of condition †, essentially it allows us to change the connection forms so that the whole of the transformed connection lies in $L^{2,1}$.

We can now take the $(0,1)$-part of (7.2) to give

$$P^{-1}\overline{\partial}P + P^{-1}\omega^{\overline{z}}P = *\overline{\partial} * \eta + *P^{-1}\overline{\partial} * bP \in L^{(2,1)}(D) \tag{7.3}$$

which after applying Theorem 3.3.1 gives us the existence of some
$Q \in L^\infty \cap W^{1,2}(D, GL(k,\mathbb{C}))$ satisfying (when $\|\omega\|_{L^2} + K$ is small enough)

$$\overline{\partial}Q = -(*\overline{\partial} * \eta + *P^{-1}\overline{\partial} * bP)Q,$$

$$\|\mathrm{dist}(\mathrm{Q},\mathrm{Id})\|_{\mathrm{L}^\infty(\mathrm{D})} \leq \frac{1}{3}$$

and

$$\|\nabla Q\|_{L^2(D_{\frac{1}{2}})} \leq C\|\omega\|_{L^2(D)}.$$

Thus we have

$$P^{-1}\overline{\partial}P + P^{-1}\omega^{\overline{z}}P = -\overline{\partial}QQ^{-1}$$

and therefore setting $S = PQ \in L^\infty(D, GL(k,\mathbb{C})) \cap W^{1,2}(D_{\frac{1}{2}}, GL(k,\mathbb{C}))$ we have

$$\overline{\partial}S = -\omega^{\overline{z}}S$$

with the desired estimates.

7.6 A few remarks

Here we explain how this chapter ties in with the two dimensional second order analogue – Rivière's equation. Here we will always denote $B_1 \subset \mathbb{R}^2$ the two dimensional unit disc.

If we go back to Rivière's observation that critical points $u \in W^{1,2}(B_1, \mathbb{R}^m)$ to conformally invariant elliptic Lagrangians solve

$$\mathrm{d}^*_\omega(\mathrm{d}u) = 0$$

for some $\omega \in L^2(B_1, so(m) \otimes \wedge^1 \mathrm{T}^*\mathbb{R}^2)$ (given explicitly in [Riv07]), one can also check that we have

$$\mathrm{d}_\omega(\mathrm{d}u) = \mathrm{d}(\mathrm{d}u) + \omega \wedge \mathrm{d}u = 0$$

as is the case for harmonic maps. These coupled PDE then allow one to be able to write

$$\overline{\partial}_{\omega^{\overline{z}}}(\partial u) = \overline{\partial}\partial u + \omega^{\overline{z}} \wedge \partial u = 0.$$

We could generalise this, and simply consider maps $v \in L^2(B_1, \mathbb{C}^m \otimes \wedge^1 \mathrm{T}^*\mathbb{R}^2)$ solving

$$\mathrm{d}^*_\omega(v) = 0$$

and

$$\mathrm{d}_\omega(v) = 0$$

for some connection $\omega \in L^2(B_1, u(m) \otimes \wedge^1 \mathrm{T}^*\mathbb{R}^2)$. As above we can check that $v^{(1,0)} \in L^2(B_1, \mathbb{C}^m \otimes \wedge^{(1,0)} \mathrm{T}^*_{\mathbb{C}}\mathbb{C})$ solves

$$\overline{\partial}_{\omega^{\overline{z}}}(v^{(1,0)}) = 0.$$

Now we can ask, under what conditions can we find a holomorphic change of frame S as in Theorem 7.2.1 in order to conclude $v \in L^\infty \cap W^{1,2}$. In general we cannot do this unless ω satisfies condition † because of the counter-example presented in section 7.3. However we are still free to change our frame via a map $P \in W^{1,2}(B_1, U(m))$, and writing $w := P^{-1} v^{(1,0)}$ we have

$$\overline{\partial}_{\omega_P^{\overline{z}}}(w) = 0$$

where

$$P^{-1}\mathrm{d}P + P^{-1}\omega P = \omega_P.$$

Now we can ask whether ω_P satisfies condition †? In particular this is the case if $\mathrm{d}(\omega_P) = 0$ (the 'opposite' of what is achieved in considering a Coulomb frame) or even $\mathrm{d}(\omega_P) \in \mathscr{H}^1$.

The bottom line here is the following:

Theorem 7.6.1. *Let $\omega \in L^2(D, \mathfrak{g}_{\mathbb{F}} \otimes \wedge^1 \mathrm{T}^*\mathbb{R}^2)$, and suppose there exists a change of frame $P \in W^{1,2}(B_1, \mathfrak{G}_{\mathbb{F}})$ such that*

$$\omega_P = P^{-1}\mathrm{d}P + P^{-1}\omega P$$

satisfies condition †. Then there exists $\varepsilon > 0$ such that whenever

$$\|\omega\|_{L^2} + K \leq \varepsilon$$

there exists a change of frame $S \in L^\infty \cap W^{1,2}(D_{\frac{2}{3}}, Gl(k, \mathbb{C}))$ such that

$$\bar{\partial} S = -\omega^{\bar{z}} S$$

with

$$\|\mathrm{dist}(\mathrm{S}, \mathrm{U(m)})\|_{\mathrm{L}^\infty(\mathrm{D}_{\frac{2}{3}})} \leq \frac{1}{3}$$

and

$$\|\nabla S\|_{L^2(D_{\frac{1}{2}})} \leq C\|\omega\|_{L^2}.$$

An interesting question here is under what circumstances one can find a change of frame such that ω_P satisfies †. This would enable a regularity theory for solutions $v \in L^2(B_1, \mathbb{C}^m \otimes \wedge^1 \mathrm{T}^*\mathbb{R}^2)$ to

$$\mathrm{d}^*_\omega(v) = 0$$

and

$$\mathrm{d}_\omega(v) = 0$$

for some connection $\omega \in L^2(B_1, u(m) \otimes \wedge^1 \mathrm{T}^*\mathbb{R}^2)$. In particular one could conclude that $v \in (L^\infty \cap W^{1,2})_{loc}$ with $|v|^2 \in h^1$.

As stated above, and letting $v = \mathrm{d}u$ we can write the PDE solved by critical points u of conformally invariant Lagrangians, in this form. Another example of such solutions are conformal immersions of a disc in $\mathbb{R}^3$ with square integrable mean curvature, again with $v = \mathrm{d}u$. In both of these examples ω is given in [Riv07]. It is therefore interesting to see whether one could obtain a holomorphic gauge S in these other geometric situations (in other words does ω or ω_P satisfy condition †). In fact we are aware that this can be done under further regularity assumptions on the Lagrangian, or in the case of conformal immersions of a disc in $\mathbb{R}^3$, one must assume that the mean curvature is

Lipschitz. In particular one can recover all the local regularity theory and also some recent results of Lamm and Lin [LL12] – we will elaborate on this further in a separate work.

Appendix A

Hodge decompositions

We denote here the Hodge star operator on k-forms by $*$ and the related exterior derivative, d and divergence operator d^*-which is the formal adjoint of the exterior derivative:

$$\mathrm{d}^* : \Gamma(\wedge^k \mathrm{T}^* \mathscr{M}) \to \Gamma(\wedge^{k-1} \mathrm{T}^* \mathscr{M})$$

satisfies

$$\mathrm{d}^* v = (-1)^{n(k+1)+1} * \mathrm{d} * v$$

for $v \in \Gamma(\wedge^k \mathrm{T}^* \mathscr{M})$. There is a natural point-wise inner product for k-forms given by $\langle \omega^1, \omega^2 \rangle = *(\omega^1 \wedge *\omega^2)$ and an L^2-inner product given by $(\omega^1, \omega^2) = \int *\langle \omega^1, \omega^2 \rangle$.

Our main reference here is [Mor66, Chapter 7] where we can find all of the results stated below, in particular we require the following.

Lemma A.0.2. *Suppose* $\omega \in L^2(B_1, \wedge^1 \mathbb{R}^n)$ *then there are unique* $a \in W_0^{1,2}(B_1)$, $b \in W_N^{1,2}(B_1, \wedge^2 \mathbb{R}^n)$ *and a harmonic one form* $h \in L^2(B_1, \wedge^1 \mathbb{R}^n)$ *such that*

$$\omega = \mathrm{d}a + \mathrm{d}^* b + h.$$

Moreover $\mathrm{d}b = 0$ *with*

$$\|a\|_{W^{1,2}(B_1)} + \|b\|_{W^{1,2}(B_1)} + \|h\|_{L^2(B_1)} \leq C\|\omega\|_{L^2(B_1)}$$

and

$$\|\mathrm{d}a\|^2_{L^2(B_1)} + \|\mathrm{d}^* b\|^2_{L^2(B_1)} + \|h\|^2_{L^2(B_1)} = \|\omega\|^2_{L^2(B_1)}.$$

In particular if either $\mathrm{d}\omega = 0$ *or* $\delta\omega = 0$ *then*

$$\omega = \mathrm{d}\hat{a} \ \textit{ and } \ \|\hat{a}\|_{W^{1,2}(B_1)} \leq C\|\omega\|_{L^2(B_1)}$$

or

$$\omega = \mathrm{d}^* \hat{b} \text{ and } \|\hat{b}\|_{W^{1,2}(B_1)} \leq C\|\omega\|_{L^2(B_1)}$$

respectively.

We note here that $W_N^{1,2}(B_1, \wedge^k\mathbb{R}^n)$ is the space of forms whose normal boundary part vanishes, which we may define in a trace sense or equivalently for any smooth $k-1$ form v we have $(\omega, \mathrm{d}v) = (\mathrm{d}^*\omega, v)$ when $\omega \in W_N^{1,2}(B_1, \wedge^k\mathbb{R}^n)$.

Otherwise we have the more general formula for smooth k and $k-1$ forms respectively (which follows from Stoke's theorem)

$$(\omega, \mathrm{d}v) = (\mathrm{d}^*\omega, v) + \int_\partial v_T \wedge *\omega_N$$

where $_T$ and $_N$ denote the tangential and normal components. (The latter holds for any appropriate Sobolev forms by approximation). Note we could easily define $W_T^{1,2}(B_1, \wedge^k\mathbb{R}^n)$ in a weak sense also. We use the following fact which follows by approximation: For $a \in W^{1,2}(B_1, \wedge^{k-1}\mathbb{R}^n)$, $b \in W^{1,2}(B_1, \wedge^k\mathbb{R}^n)$ we have $(\mathrm{d}a, \mathrm{d}^*b) = 0$ if either $a_T = 0$ or $b_N = 0$.

Note that we have $\Delta_{LB} a = \mathrm{d}^*\omega$ and $\Delta_{LB} b = \mathrm{d}\omega$ in a weak sense since $\mathrm{d}h = \mathrm{d}^*h = 0$, $\mathrm{d}b = 0$ and since a is a function ($\Delta_{LB} = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$). Since B_1 is Euclidean we have $\Delta_{LB} u = -\Delta u$ where Δ is the standard Laplacian on functions.

In the case that $n = 2$ we sometimes use different notation: in the Hodge decomposition of Lemma A.0.2 we see that $b_N = b$, thus $b \in W_0^{1,2}(B_1, \wedge^2\mathbb{R}^2)$. Setting $\hat{b} = *b \in W_0^{1,2}(B_1)$ we have $\mathrm{d}^*b = *\mathrm{d}\hat{b}$ and we may write

$$\omega = \nabla\hat{a} + \nabla^\perp\hat{b}$$

where $\nabla = (\partial_x, \partial_y)$ is the gradient and $\nabla^\perp = (-\partial_y, \partial_x)$ (or $*\mathrm{d}$) is its rotation by $\frac{\pi}{2}$. We have also written $h = \mathrm{d}\tilde{a}$ and $\hat{a} = a + \tilde{a}$.

Appendix B

Scaling

Lemma B.0.3. *Suppose $f \in L\log L(B_r(x_0))$ and $r \in (0, 1/2]$. Then there exists $C < \infty$ such that*

$$\|f\|_{L^1(B_r(x_0))} \le C\left[\log\left(\frac{1}{r}\right)\right]^{-1} \|f\|_{L\log L(B_r(x_0))}.$$

Proof. Notice that

$$\begin{aligned}
0 &\le r^2 \int_0^{|B_1|} f^*(r^2 t) \log\left(e + \frac{1}{t}\right) dt \\
&= \int_0^{|B_r(x_0)|} f^*(s) \log\left(e + \frac{r^2}{s}\right) ds \\
&= \int_0^{|B_r(x_0)|} f^*(s) \log r^2 \, ds + \int_0^{|B_r(x_0)|} f^*(s) \log\left(\frac{e}{r^2} + \frac{1}{s}\right) ds \\
&\le -2\log\left(\frac{1}{r}\right) \|f\|_{L^1(B_r(x_0))} + C\|f\|_{L\log L(B_r(x_0))},
\end{aligned}$$

where the final inequality is obtained by noticing that $s \le \pi r^2 < 1$ which implies $\frac{e}{r^2} + \frac{1}{s} \le \frac{e\pi+1}{s} \le \left(e + \frac{1}{s}\right)^C$ for some fixed C. $\square$

The following lemma indicates that $L\log L$ norms do not deteriorate under scaling. However we emphasise that they need not improve, unlike L^p norms for $p > 1$.

Lemma B.0.4. *Suppose $f \in L\log L(B_r(x_0))$ where $r \in (0, 1/2]$. Defining $\hat{f} := r^2 f(x_0 + rx)$ there exists $C < \infty$ such that*

$$\|\hat{f}\|_{L\log L(B_1)} \le C\|f\|_{L\log L(B_r(x_0))}.$$

Proof. First we calculate

$$\begin{aligned}
\hat{f}^*(t) &= \inf\{s \geq 0 : |\{x \in B_1 : |\hat{f}(x)| > s\}| \leq t\} \\
&= \inf\{s \geq 0 : |\{x \in B_1 : |r^2 f(x_0 + rx)| > s\}| \leq t\} \\
&= \inf\{s \geq 0 : |\{y \in B_r(x_0) : |f(y)| > \frac{s}{r^2}\}| r^{-2} \leq t\} \\
&= r^2 f^*(r^2 t)
\end{aligned}$$

therefore

$$\begin{aligned}
\|\hat{f}\|_{L\log L(B_1)} &\leq C\int_0^{|B_1|} \hat{f}^*(t) \log\left(e + \frac{1}{t}\right) dt \\
&= C\int_0^{|B_1|} r^2 f^*(r^2 t) \log\left(e + \frac{1}{t}\right) dt \\
&= C\int_0^{|B_r(x_0)|} f^*(s) \log\left(e + \frac{r^2}{s}\right) ds \\
&\leq C\|f\|_{L\log L(B_r(x_0))}.
\end{aligned}$$

□

We will need to consider u, ω and f solving

$$-\Delta u = \omega.\nabla u + f$$

on some small ball $B_R(x_0) \subset B_1$. In order to do so we re-scale $\hat{u}(x) := u(x_0 + Rx)$, $\hat{\omega}(x) := R\omega(x_0 + Rx)$ and $\hat{f} := R^2 f(x_0 + Rx)$. First of all we see that

$$-\Delta \hat{u} = \hat{\omega}.\nabla \hat{u} + \hat{f}$$

on B_1 and we list the scaling properties of the related norms as follows.

1. $\|\hat{\omega}\|_{M^{2,n-2}(B_1)} = \|\omega\|_{M^{2,n-2}(B_R(x_0))}$.
2. $[\hat{u}]_{C^{0,\gamma}(B_1)} = R^\gamma [u]_{C^{0,\gamma}(B_R(x_0))}$.
3. $\|\hat{u}\|_{L^1(B_1)} = R^{-n}\|u\|_{L^1(B_R(x_0))}$.
4. $\|\nabla \hat{u}\|_{M^{l,\nu}(B_1)} = R^{\frac{l-(n-\nu)}{l}}\|\nabla u\|_{M^{l,\nu}(B_R(x_0))}$.
5. Setting $\nu = 0$ above gives $\|\nabla \hat{u}\|_{L^l(B_1)} = R^{1-\frac{n}{l}}\|\nabla u\|_{L^l(B_R(x_0))}$.
6. We also have that the Lorentz spaces $L^{(l,\infty)}$ or 'weak'-L^l scale in the same fashion

as the usual L^l spaces,

$$\|\nabla \hat{u}\|_{L^{(l,\infty)}(B_1)} = R^{1-\frac{n}{l}} \|\nabla u\|_{L^{(l,\infty)}(B_R(x_0))}.$$

7. $\|\hat{f}\|_{L^p(B_1)} = R^{2-\frac{n}{p}} \|f\|_{L^p(B_R(x_0))}$.

8. For $f \in L^p(B_1)$ and $1 \le s \le p$ we have

$$\|f\|_{M^{s,n(1-\frac{s}{p})}(B_1)} \le C\|f\|_{L^p(B_1)}$$

for $C = C(n,p)$, since by Hölder's inequality

$$\begin{aligned} \|f\|^s_{L^s(B_R)} &\le CR^{n(\frac{1}{s}-\frac{1}{p})s}\|f\|^s_{L^p(B_1)} \\ &= CR^{n(1-\frac{s}{p})}\|f\|^s_{L^p(B_1)}. \end{aligned}$$

9. $\|\hat{f}\|_{L\log L(B_1)} \le C\|f\|_{L\log L(B_R(x_0))}$

where the final estimate is following from Lemma B.0.4.

Appendix C

Absorption lemma

Special cases of the following lemma are widely used in regularity theory.

Lemma C.0.5. *(Leon Simon [Sim96, §2.8, Lemma 2].) Let* $B_\rho(y) \subset \mathbb{R}^n$ *be any ball,* $k \in \mathbb{R}$, $\Gamma > 0$*, and let* φ *be any* $[0, \infty)$*-valued convex sub-additive function on the collection of convex subsets of* $B_\rho(y)$*; thus* $\varphi(A) \leq \sum_{j=1}^N \varphi(A_j)$ *whenever* $A, A_1, A_2, \ldots, A_N$ *are convex subsets of* $B_\rho(y)$ *with* $A \subset \bigcup_{j=1}^N A_j$*. There is* $\epsilon_0 = \epsilon_0(k, n)$ *such that if*

$$\sigma^k \varphi(B_{\sigma/2}(z)) \leq \epsilon_0 \sigma^k \varphi(B_\sigma(z)) + \Gamma$$

whenever $B_{2\sigma}(z) \subset B_\rho(y)$*, then there exists some* $C = C(k, n) < \infty$ *such that*

$$\rho^k \varphi(B_{\rho/2}(y)) \leq C\Gamma.$$

In particular we can apply this lemma when $\varphi(A) = \|k\|^p_{M^{p,\beta}(A)}$.

Appendix D

Weak convergence of measures and functions of bounded variation

We consider the space of functions of bounded variation $BV(B)$ for any ball $B \subset \mathbb{R}^2$. BV is defined by $BV(B) = \{V \in L^1(B) : \int_B |\nabla V| := \sup_{\phi \in C_0^1(B,\mathbb{R}^2)\ \|\phi\|_{L^\infty} \leq 1} \int_B V \mathrm{div}\phi < \infty\}$. In other words it is the space of functions whose distributional derivatives are signed Radon measures with finite total mass. This is a Banach space with norm $\|V\|_{BV(B)} = \|V\|_{L^1(B)} + \int_B |\nabla V|$. It is easy to see that we have the continuous embedding $W^{1,1} \hookrightarrow BV$, moreover in two dimensions we have the continuous embedding $BV(B) \hookrightarrow L^2(B)$ and the compact embeddings $BV(B) \hookrightarrow L^p(B)$ for any $p < 2$ (see for instance [Zie89]).

We also use the standard weak-$*$ compactness available in the space of signed Radon measures with finite total mass, denoted M.

The proof of the next lemma is essentially taken from [Eva90, Theorem 9] and is similar to that stated in [LZ09]. For an integrable function k we implicitly view it as both a function and a measure, i.e. $k = k\,dx$.

Lemma D.0.6. *Suppose $\{V_n\} \subset BV(B)$ is a bounded sequence and $B \subset \mathbb{R}^2$ is an open ball. Then there exist at most countable $\{x_j\} \subset B$ and $\{a_j > 0\}$ (where $\sum_j a_j < \infty$) and $V \in BV(B)$ such that (up to a subsequence)*

$$V_n^2 \rightharpoonup V^2 + \sum_j a_j \delta_{x_j}$$

weakly in $M(B)$.

Proof. Since $\{V_n\} \subset BV(B)$ is a bounded sequence, there exists $V \in L^2$ such that (up to a subsequence) $V_n \to V$ strongly in L^p for all $p < 2$ and $V_n \rightharpoonup V$ weakly in L^2. Also $\{\nabla V_n\} \subset M(B)$ is bounded so (again up to a subsequence) $\nabla V_n \rightharpoonup \lambda$ (a vector-valued measure)$\in M(B)$. In particular, for all $\phi \in C_c^1(B, \mathbb{R}^2)$

$$\begin{aligned}\int \phi . d\lambda &= \lim_{n\to\infty} \int \phi . \nabla V_n \, dx \\ &= -\lim_{n\to\infty} \int \operatorname{div}(\phi) V_n \, dx \\ &= -\int \operatorname{div}(\phi) V \, dx.\end{aligned}$$

In other words $V \in BV(B)$ and $\nabla V = \lambda$.

Now set $g_n := V_n - V$. Note that $|\nabla g_n| \in M(B)$ is bounded so for a subsequence $|\nabla g_n| \rightharpoonup \mu \in M(B)$ where μ is non-negative. Similarly (up to a subsequence) $g_n^2 \rightharpoonup \nu \in M(B)$ where ν is also non-negative. We have that for all $\phi \in C_c^1(B)$, $\phi g_n \in BV(B)$ and by the continuous embedding $BV(B) \hookrightarrow L^2(B)$ we have

$$\left(\int (\phi g_n)^2 \, dx\right)^{1/2} \leq C \int |\nabla(\phi g_n)| \, dx$$

and since $g_n \to 0$ in L^1, taking limits gives

$$\left(\int \phi^2 \, d\nu\right)^{1/2} \leq C \int |\phi| \, d\mu.$$

Taking ϕ to be an approximation to the characteristic function on $B_r(x) \subset B$ we get

$$\nu(B_r(x)) \leq C(\mu(B_r(x)))^2$$

for all $B_r(x) \subset B$, and in particular $\nu \ll \mu$.

By standard results for differentiation of measures (see e.g. [EG92, §1.6 Theorem 2]), for any Borel set $E \subset B$

$$\nu(E) = \int_E D_\mu \nu \, d\mu$$

where $D_\mu \nu = \lim_{r\downarrow 0} \frac{\nu(B_r(x))}{\mu(B_r(x))}$ is a μ-integrable function (this limit exists μ-almost everywhere).

Since μ is a finite, positive Radon measure, there are at most countable points $\{x_j\}$ such that $\mu(\{x_j\}) > 0$, and if $\mu(\{x\}) = 0$ then

$$D_\mu \nu(x) = \lim_{r\downarrow 0} \frac{\nu(B_r(x))}{\mu(B_r(x))} \leq C \lim_{r\downarrow 0} \mu(B_r(x)) = 0.$$

Letting $X := \cup_j \{x_j\}$ we have $D_\mu \nu = 0$ μ-almost everywhere on $B \backslash X$. Hence $D_\mu \nu$ is a simple function, therefore for Borel $E \subset B$

$$\nu(E) = \int_E D_\mu \nu \, d\mu = \sum_{\{j : x_j \in E\}} D_\mu \nu(x_j) \mu(\{x_j\}).$$

Setting $a_j := D_\mu \nu(x_j)\mu(\{x_j\})$ we have $\nu = \sum_j a_j \delta_{x_j}$. Now, for $\phi \in C_c^0(B)$

$$\begin{aligned}
\sum_j a_j \phi(x_j) &= \lim_{n\to\infty} \int g_n^2 \phi \, dx \\
&= \lim_{n\to\infty} \int (V_n - V)^2 \phi \, dx \\
&= \lim_{n\to\infty} \left(\int (V_n^2 - V^2)\phi \, dx + 2 \int V(V - V_n)\phi \, dx \right)
\end{aligned}$$

where the last term vanishes in the limit since $V_n \rightharpoonup V$ weakly in L^2. □

Lemma D.0.7 (Corollary of Lemma D.0.6)**.** *Suppose* $\{V_n\}$ *is as in Lemma D.0.6. If*

$$\lim_{r\downarrow 0} \limsup_{n\to\infty} \|V_n\|_{L^2(B_r(x))} = 0$$

for all $x \in B$*, then*

$$V_n \to V$$

strongly in $L^2_{loc}(B)$ *(same* V *as in Lemma D.0.6).*

Proof. First we apply Lemma D.0.6 and viewing $|V_n|^2 dx$ as a sequence in $M(B)$ we notice that the condition $\lim_{r\downarrow 0} \limsup_{n\to\infty} \|V_n\|_{L^2(B_r(x))} = 0$ simply says that $V_n^2 \rightharpoonup V^2$ weakly in $M(B)$. Therefore, given any open ball $B_r(x) \subset\subset B$ we can apply standard results for Radon measures ([EG92, §1.9 Theorem 1]) to conclude that (since $\int_{\partial B_r(x)} |V|^2 dx = 0$) $\|V_n\|_{L^2(B_r(x))} \to \|V\|_{L^2(B_r(x))}$ for all $B_r(x) \subset\subset B$. Hence

$$\begin{aligned}
\int_{B_r(x)} (V - V_n)^2 \, dx &= \int_{B_r(x)} (V_n^2 - V^2) dx + 2 \int_{B_r(x)} V(V - V_n) \, dx \\
&\to 0 \quad \text{as } n \to \infty
\end{aligned}$$

since $V_n \rightharpoonup V$ weakly in L^2. Therefore $V_n \to V$ strongly in $L^2_{loc}(B)$. □

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